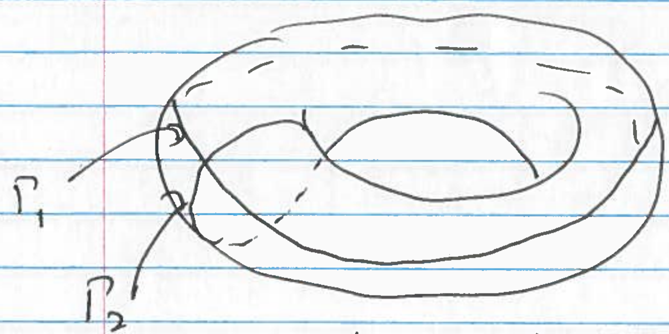
L 25 11/17/2022 (242)

Vacuum Degeneracy on a torus



$M = T^2 \times \mathbb{R}$
↑ torus

On a torus the flat connections have global degrees of freedom.
(holonomies = non-contractible loops)

Wilson loops on the holonomies of T^2

$$\int_0^{L_1} dx_1\, a_1 = \oint_{\Gamma_1} dx_\mu\, a^\mu \equiv \bar{a}_1$$

$$\oint_{\Gamma_2} dx_\mu\, a^\mu = \int_0^{L_2} dx_2\, a_2 \equiv \bar{a}_2$$

gauge invariant and time dependent

"large" gauge transf

$$\Rightarrow a_1 = \partial_1 \Phi + \frac{\bar{a}_1}{L_1}, \quad a_2 = \partial_2 \Phi + \frac{\bar{a}_2}{L_2}$$

"small" gauge transf.

<u>Action:</u> $S = \frac{k}{4\pi} \int dx_0\, \varepsilon_{ij}\, \bar{a}_i \partial_0 \bar{a}_j$

$\Rightarrow$ finite quantum system!

with $[\bar{a}_1, \bar{a}_2] = i\,\frac{2\pi}{k} \Rightarrow \bar{a}_2 = -i\,\frac{2\pi}{k}\frac{\partial}{\partial \bar{a}_1}$

Wilson loops along Γ_1 and Γ_2

$$W[\Gamma_1] = \exp\left(i\int_0^{L_1} dx_1\, a_1\right) = e^{i\bar{a}_1}$$

$$W[\Gamma_2] = \exp\left(i\int_0^{L_2} dx_2\, a_2\right) = e^{i\bar{a}_2}$$

$$W[\Gamma_1]\, W[\Gamma_2] = e^{-i\frac{2\pi}{k}}\, W[\Gamma_2]\, W[\Gamma_1]$$

Under large gauge transf.

$$\bar{a}_1 \to \bar{a}_1 + 2\pi \quad , \quad \bar{a}_2 \to \bar{a}_2 + 2\pi$$

$W[\Gamma_1]$ and $W[\Gamma_2]$ are invariant

$\Rightarrow$ $\bar{a}_1$ and $\bar{a}_2$ define a ~~tor~~ 2-torus target space.

Let $U_1 = e^{ik\bar{a}_2}$ and $U_2 = e^{-ik\bar{a}_1}$

s.t. $U_1 U_2 = e^{i2\pi k}\, U_2 U_1$ (commute if $k \in \mathbb{Z}$)

$$U_1^{-1}\, W[\Gamma_1]\, U_1 = W[\Gamma_1]$$

$$U_2^{-1}\, W[\Gamma_2]\, U_2 = W[\Gamma_2]$$

Let $|0\rangle$ be an eigenstate of $W[\Gamma_1]$ with

e.v. 1 $\Leftrightarrow$ $W[\Gamma_1]|0\rangle = |0\rangle$

$$\Rightarrow \quad W[\Gamma_1]\, W[\Gamma_2]|0\rangle = e^{-i\frac{2\pi}{k}}\, W[\Gamma_2]W[\Gamma_1]|0\rangle$$

$$= e^{-i\frac{2\pi}{k}}\, W[\Gamma_2]$$

$\Rightarrow W[\Gamma_2]|0\rangle$ is an eigenstate of $W[\Gamma_1]$

with e.v. $e^{-i2\pi/k}$

~~[illegible]~~ More generally

$$W[\Gamma_1]\, W[\Gamma_2]^p |0\rangle = e^{-i\frac{2\pi}{k}p}\, W[\Gamma_2]^p |0\rangle$$

eigenvalue!

If $k \in \mathbb{Z} \Rightarrow$ there are $p = 0, 1, 2, \ldots, k-1$

eigenstates

$\Rightarrow$ The vacuum state is k-fold degenerate

on a 2-torus

More generally on a manifold with

g handles ($g \in \mathbb{Z}$) the degeneracy is k^g

Sphere: $g = 0 \Rightarrow$ unique state

torus: $g = 1 \Rightarrow k$ states

etc.

The theory $\mathcal{L} = \frac{k}{4\pi} \varepsilon_{\mu\nu\lambda} a^\mu \partial^\nu a^\lambda$

is called the $U(1)_k$ Chern-Simons theory

Non abelian: $SU(2)_k$

Chern-Simons and Fractional Statistics

Consider the $U(1)_k$ Chern-Simons theory and two Wilson loops on γ_1 and γ_2 (closed paths)

$$W[\gamma_1 \cup \gamma_2] = \left\langle e^{i\oint_{\gamma_1\cup\gamma_2} dx_\mu a^\mu} \right\rangle_{CS}$$

union!

$$\langle e^{i\int_\gamma a} \rangle \equiv \left\langle e^{i\int d^3x\, J_\mu a^\mu} \right\rangle_{CS}$$

$$J_\mu(x) = \delta(x_\mu - z_\mu(t)) \frac{dz_\mu}{dt}$$ ← tangent unit vector

$z_\mu(t)$ is a parametrization of the loop γ.

$$\left\langle e^{i\oint_\gamma dx_\mu a^\mu} \right\rangle \equiv e^{iI[\gamma]_{CS}}$$

$$= e^{-\frac{i}{2}\int d^3x \int d^3y\, J_\mu(x)\, G^{\mu\nu}(x-y)\, J_\nu(y)}$$

$$G^{\mu\nu}(x-y) = \langle T\, a^\mu(x)\, a^\nu(y) \rangle_{CS}$$

and $\partial_\mu J^\mu = 0$ (since γ is a set of closed loops)

In the Euclidean metric (imaginary time)

$$G_{\mu\nu}(x-y) = \frac{2\pi}{k} \varepsilon_{\mu\nu\lambda} \partial_\lambda G_0(x-y)$$

where $-\partial^2 G_0(x-y) = \delta(x-y)$

$$G_0(x-y) = \langle x| -\frac{1}{\partial^2} |y\rangle = \frac{1}{4\pi|x-y|}$$

$$\Rightarrow I_{CS}(\gamma) = \frac{\pi}{k} \int d^3x \int d^3y \, J_\mu(x) \, \varepsilon_{\mu\nu\lambda} \partial_\lambda G_0(x-y) J_\nu(y)$$

$$= \frac{\pi}{k} \oint_\gamma dx_\mu \oint_\gamma dy_\nu \, \varepsilon_{\mu\nu\lambda} \partial_\lambda G_0(x-y)$$

Since J_μ is conserved $\Rightarrow \partial_\mu J_\mu = 0$

$$\Rightarrow J_\mu = \varepsilon_{\mu\nu\lambda} \partial_\nu B_\lambda \qquad \text{(Faraday)}$$

$$\partial_\lambda B_\lambda = 0 \qquad \text{(Lorentz gauge)}$$

As in magnetostatics

$$B_\lambda = \varepsilon_{\mu\nu\lambda} \partial_\nu \phi_\lambda \Rightarrow J_\mu(x) = -\partial^2 \phi_\lambda(x)$$

$$\phi_\lambda(x) = \int d^3y \, G_0(x-y) J_\mu(y)$$

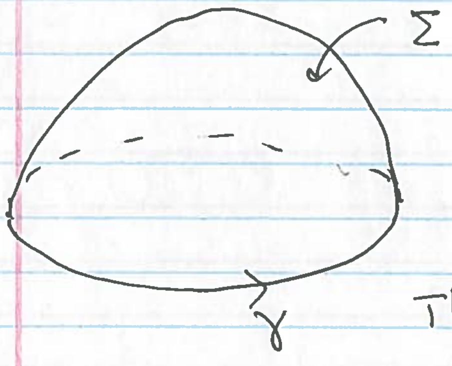
$$\Rightarrow B_\lambda(x) = \varepsilon_{\mu\nu\lambda} \partial_\nu \phi_\lambda(x) = \int d^3y \, \varepsilon_{\mu\nu\lambda} \partial_\lambda G_0(x-y) J_\mu(y)$$

$$\Rightarrow I_{CS}(\gamma) = \frac{\pi}{k} \int d^3x \, J_\mu(x) \, B_\mu(x)$$

where $B_\mu(x)$ is the field created by $J_\mu(x)$!

Using Stokes Theorem

$$I_{CS}(\gamma) = \frac{\pi}{k} \oint_\gamma dx_\mu \, B_\mu(x) = \frac{\pi}{k} \int_\Sigma dS_\mu \, \varepsilon_{\mu\nu\lambda} \partial_\nu B_\lambda$$



$$\Rightarrow I_{CS}(\gamma) = \frac{\pi}{k} \int_\Sigma dS_\mu \, J_\mu$$

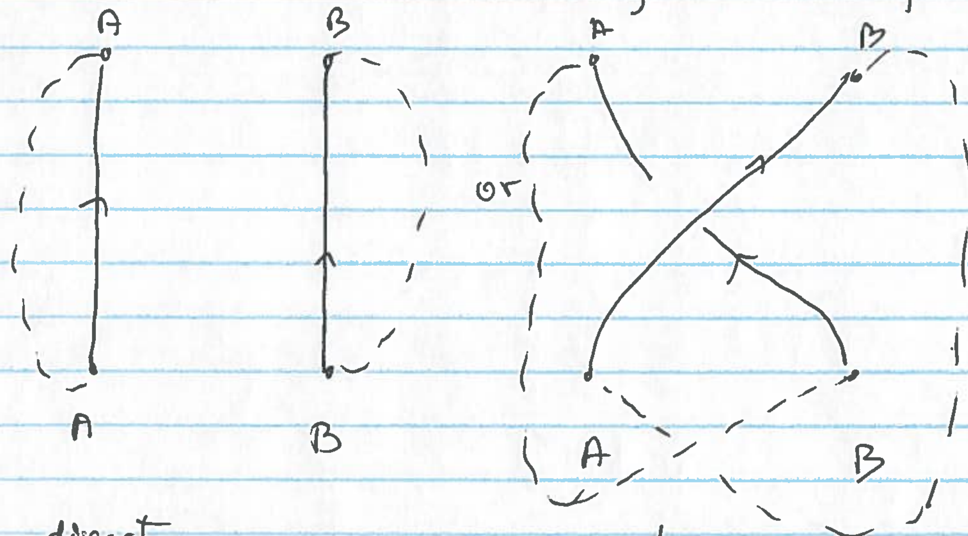
The integral counts the number of times n_γ that the Wilson loop pierces the surface Σ $\Rightarrow$ it is an integer called the <u>linking number</u> or <u>Gauss invariant</u>

$$\Rightarrow \langle W[\gamma] \rangle_{CS} = e^{i \frac{2\pi}{k} n_\gamma}$$

linking number (topological invariant)

How is this related to fractional statistics?

Consider two heavy particles coupled to a CS gauge field s.t. at $T \to -\infty$ they are located at points A and B and as $T \to +\infty$ they switch places.



direct
unlinked

exchange
linked (half braid)

$\Rightarrow$ the world lines are braided under an exchange process $\Rightarrow$ non-trivial knot
(paths cannot cross!)

$\Rightarrow W_{direct}$, $W_{exchange}$ ← half braid

$$W_{exchange} = W_{direct}\, e^{\pm i\pi/k}$$

$$\Rightarrow \Psi[B,A] = \Psi(A,B)\, e^{\pm i\pi/k}$$

⇒ particles coupled to a CS gauge field are anyons.

If $k=1$ ⇒ fermions

$k \to \infty$ ⇒ bosons

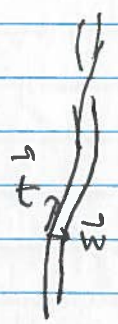
$k=2$ ⇒ "semions" etc.

How about a single Wilson loop?

⇒ self-linking # (~~[illegible]~~ writhing #)

⇒ extend the worldline to a narrow ribbon

We can now define a "frame"



with $\vec{t}$ the tangent vector, $\vec{w}$ → and $\vec{w} \times \vec{t}$

This frame can precess along the worldribbon ⇒ thus defines a self-winding

⇒ topological spin (consistent with spin-statistics)

$$\langle e^{i\oint_\gamma a} \rangle_{CS} = e^{\pm i\frac{\pi}{k}}$$

is the spin

Useful identity (valid on a disk)

$$\mathcal{L} = \frac{k}{4\pi} \varepsilon_{\mu\nu\lambda} a^\mu \partial^\nu a^\lambda + \frac{1}{2\pi} \varepsilon_{\mu\nu\lambda} a^\mu \partial^\nu b^\lambda$$

$\Rightarrow$ EOM $\quad \dfrac{\delta \mathcal{L}}{\delta a_\mu} = 0$

$$\Rightarrow \quad \frac{k}{2\pi} \varepsilon_{\mu\nu\lambda} \partial^\nu a^\lambda + \frac{1}{2\pi} \varepsilon_{\mu\nu\lambda} \partial^\nu b^\lambda = 0$$

$\Rightarrow \quad k\, a_\mu = - b_\mu$ (up to a gauge transf.)

Plugging back in

$$\mathcal{L}(b) = -\frac{1}{k} \frac{1}{4\pi} \varepsilon_{\mu\nu\lambda} b^\mu \partial^\nu b^\lambda$$

"duality" $\quad k \leftrightarrow -\frac{1}{k}$

The statistical phase now is $e^{\pm i\pi k}$

$\Rightarrow$ If k is an even integer $e^{\pm i\pi k} = 1$

k is an odd integer $e^{\pm i\pi k} = -1$

We can use this result for a

theory of statistical transmutation!