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We will now use these ideas in the context of the FQHE. Consider a system of N electrons on a plane with a magnetic field B and e-e interactions. In first quantization the Hamiltonian is (fully polarized)

$$H = \sum_{i=1}^{N} \left\{ \frac{1}{2M} \left(\vec{p}_j - \frac{e}{c} \vec{A}(\vec{x}_j) \right)^2 + e A_0 (x_j) \right\}$$

$$+ \sum_{i<j} V(|\vec{x}_i - \vec{x}_j|)$$

In second quantization we have the action for the electron field ($d^3z \equiv d^2z \, dz_0$) time

$$S = \int d^3z \left\{ \psi^*(z) \left(i D_0 + \mu \right) \psi(z) + \frac{\hbar^2}{2M} |\vec{D} \psi(z)|^2 \right\}$$

$$- \frac{1}{2} \int d^3z \int d^3z' \left(|\psi(z)|^2 - \rho_0 \right) V(z - z') \left(|\psi(z')|^2 - \rho_0 \right)$$

$$V(z - z') = V(\vec{z} - \vec{z}') \, \delta(z_0 - z_0') \quad \text{(instantaneous)}$$

$$Z = \int \mathcal{D}\psi^* \mathcal{D}\psi \, e^{iS} \; ; \; D_\mu = \partial_\mu + i \frac{e}{\hbar c} A_\mu$$

The time evolution in first quantization is a sum over the histories (world lines) of the electrons.

In the non-relativistic case the worldlines evolve from $T=-\infty$ to $T=+\infty$ but do not close. If the interactions are strong the worldlines do not cross. Direct and exchange processes are half braids of the worldlines with a weight (-1) per each exchange (half braid)

We can now construct an equivalent theory of bosons if we couple the electrons to a gauge field such that each half braid has an extra (-1). These are the composite ~~Alternatively we can map the~~ bosons.

$(\hbar=1)$

$$S_B = \int d^3z \left\{ \phi^*(z)\,(iD_0+\mu)\,\phi(z) + \frac{1}{2M}\left|\vec{D}\phi\right|^2 \right\}$$

bosons!

$$-\frac{1}{2}\int_{z,z'} \left(|\phi(z)|^2-\rho_0\right) V(z-z') \left(|\phi(z')|^2-\rho_0\right)$$

$$+\int d^3x \left[\frac{1}{2\pi}\,\varepsilon_{\mu\nu\lambda}\, a^\mu \partial^\nu b^\lambda + \frac{k}{4\pi}\,\varepsilon_{\mu\nu\lambda}\, b^\mu \partial^\nu b^\lambda\right]$$

$$D_\mu = \partial_\mu + i\frac{e}{c}A_\mu + i a_\mu \;;\quad \text{with} \quad k=2n+1 \ \text{odd}$$

Landau - Ginzburg Theory

(Zhang, Hansson, Kivelson '89)

$$Z = \int \mathcal{D}\phi^* \mathcal{D}\phi\, \mathcal{D}a_\mu \mathcal{D}b_\mu \exp\left(i S(\phi, \phi^*, a_\mu, b_\mu)\right)$$

They included a local repulsive interaction

$$\sim \lambda |\phi|^4$$

Integrating out $b_\mu \Rightarrow$ action for a_μ

becomes $+ \frac{1}{4\pi k} \varepsilon_{\mu\nu\lambda} a^\mu \partial^\nu a^\lambda$

Equations of motion

local repulsion

$$\frac{\delta}{\delta\phi^*} \longrightarrow (iD_0 + \mu)\phi - \frac{1}{2M}\vec{D}^2\phi - 2\lambda|\phi|^2\phi$$

$$- \phi \int_{x'} V(x - x')\left(|\phi(x')|^2 - \rho_0\right) = 0$$

$$\frac{\delta}{\delta a_0} \qquad \frac{1}{2\pi k}\langle \mathcal{B} \rangle + |\phi|^2 = 0$$

$$\mathcal{B} = \partial \wedge a$$

&

$$\frac{\delta}{\delta a_j} \qquad \frac{1}{2\pi k} \varepsilon_{i\alpha\beta}\, \partial^\alpha a^\beta + \frac{i}{M}\left(\phi^* D_i \phi - (D_i\phi)^* \phi\right) = 0$$

$$\frac{\delta}{\delta \mu} \qquad \int_x |\phi(x)|^2 = \rho^2 L^2 T$$

$$D_0 = \partial_0 - i(A_0 + a_0)$$

$$\vec{D} = \vec{\nabla} - i(\vec{A} + \vec{a})$$

<u>uniform solution</u>

$$|\phi|^2 = \rho_0$$

$$\rho_0 + \frac{1}{2\pi k}\langle B\rangle = 0$$

$$\mu - 2\lambda\rho_0 = 0$$

$$\langle a_\mu\rangle + A_\mu = 0$$

filling fraction

$$\rho_0 - \frac{\nu}{2\pi l_0^2} = 0 \quad , l_0 = \frac{1}{\sqrt{B}}$$

$$\Rightarrow \langle B\rangle = -B \quad (\text{screening})$$

$$\Rightarrow \cancel{N_B}\ N_e + \frac{1}{2\pi k}\langle B\rangle L^2 = 0$$

$$N_e - \frac{1}{2\pi k} B L^2 = 0$$

$$BL^2 = 2\pi N_\phi$$

$$N_e - \frac{1}{2\pi k} 2\pi N_\phi = 0$$

$$\Rightarrow \nu = \frac{N_e}{N_\phi} = \frac{1}{k} \quad \text{with } \underline{k \text{ odd}}$$

Laughlin fractions

Low-energy fluctuations

$$\phi(z) = \sqrt{\rho_0 + \delta\rho}\; e^{i\omega(z)}$$

$$A_\mu + a_\mu = \delta a_\mu \quad (\text{since } A_\mu + \langle a_\mu \rangle = 0)$$

The action becomes

$$S_{eff}(\delta\rho, \delta a_\mu, \mu) =$$

$$= \int d^3x \Big\{ \sqrt{\rho_0 + \delta\rho}\, e^{-i\omega} (i\partial_0 + \delta a_0 + \mu) \sqrt{\rho_0 + \delta\rho}\, e^{i\omega}$$

$$- \frac{1}{2M} \left| i\vec{\nabla}\left(\sqrt{\rho_0 + \delta\rho}\, e^{i\omega}\right) + \delta\vec{a}\, \sqrt{\rho_0 + \delta\rho}\, e^{i\omega} \right|^2$$

$$- \lambda(\rho_0 + \delta\rho_0)^2 \Big\}$$

$$- \frac{1}{2} \int dt \int d^2x \int d^2x'\; \delta\rho(x)\, V(\vec{x} - \vec{x}')\, \delta\rho(x')$$

$$+ \int d^3x' \frac{1}{4\pi k} \delta a_\mu\, \varepsilon^{\mu\nu\lambda}\, \partial_\nu\, \delta a_\lambda \Big\}$$

Integrating out $\delta\rho$

$$\mathcal{L}_{eff} = \frac{\kappa}{2}(\partial_0\omega - \delta a_0)^2 - \frac{\rho_s}{2}\left(\vec{\nabla}\omega - \delta\vec{a}\right)^2$$

$$+ \frac{1}{4\pi k} \varepsilon_{\mu\nu\lambda}\, a^\mu \partial^\nu a^\lambda$$

$$\kappa = \text{compressibility} = \frac{1}{2\lambda + \frac{\tilde{V}}{4M\rho_0}}$$

$$\rho_s = \frac{\rho_0}{M} = \frac{1}{2\pi k}\frac{B}{M} = \frac{\nu}{2\pi}\hbar\omega_c \quad (\text{superfluid density})$$

Hall conductance (δA_μ vs a probe field)

$$\mathcal{L}_{eff} = \frac{\kappa}{2}\left(\partial_0 \omega + \delta a_0 + e\,\delta A_0\right)^2 - \frac{\rho_s}{2}\left(\vec{\nabla}\omega + \delta\vec{a} + e\,\delta\vec{A}\right) + \frac{1}{4\pi k}\varepsilon_{\mu\nu\lambda}\,\delta a^\mu \partial^\nu \delta a^\lambda$$

London gauge $\omega = 0$

$$\mathcal{L} = \frac{\kappa}{2}\left(\delta a_0 + e\,\delta A_0\right) - \frac{\rho_s}{2}\left(\delta\vec{a} + e\,\delta\vec{A}\right)^2 + \frac{1}{4\pi k}\varepsilon_{\mu\nu\lambda}\,\delta a^\mu \partial^\nu \delta a^\lambda$$

Integrating out δa_μ

$$\mathcal{L}_{eff}(\delta A_\mu) = \frac{e^2}{4\pi k}\varepsilon_{\mu\nu\lambda}\,\delta A^\mu \partial^\nu \delta A^\lambda$$

Induced current: $\delta J_\mu = \frac{\delta \mathcal{L}}{\delta(\delta A_\mu)} = \frac{e^2}{2\pi k}\varepsilon_{\mu\nu\lambda}\partial^\nu \delta A^\lambda$

$$\Rightarrow J_i = \frac{1}{2\pi k} e^2 \varepsilon_{ij} E_j$$

$$\Rightarrow \sigma_{xy} = \frac{1}{2\pi}\frac{e^2}{k} \Rightarrow \text{FQHE!}$$

Vortices

$$\lim_{|x|\to\infty} \phi(\vec{x}) = \sqrt{\rho_0}\, e^{i\varphi(\vec{x})}$$

$$\delta a_0 = 0 \quad \text{and} \quad \lim_{|x|\to\infty} \delta a_i = \pm \nabla_i \varphi = \pm \varepsilon_{ij} \frac{x_j}{|x|^2}$$

where $\varphi(\vec{x}) = \tan^{-1}\left(\frac{x_2}{x_1}\right)$ (azimuthal angle)

This solution satisfies

$$\lim_{|x|\to\infty} |(i\vec{\nabla} - \vec{a})\,\phi|^2 = 0$$

$\Rightarrow$ $\vec{a}$ is a pure gauge as $|x| \to \infty$

$$\vec{a} \to \vec{\nabla}\omega \quad \text{and} \quad \oint_{\Gamma} d\vec{x}\cdot\vec{a} = \pm 2\pi$$

Γ large circle

We want to determine the charge and statistics of a vortex.

Charge density: $J_0 = -\dfrac{\delta S}{\delta A_0} = +\dfrac{\delta S_{cs}}{\delta a_0}$ (probe)

$$= \frac{1}{2\pi k} \varepsilon_{ij} \partial_i a_j$$

$$\Rightarrow Q = e\int d^2x\, J_0 = \frac{e}{2\pi k} \int d^2x\, \varepsilon_{ij}\partial_i a_j = \frac{e}{2\pi k} \oint dx_i\, a_i = \pm\frac{e}{k}$$

$$\Rightarrow Q = \pm e/k$$

To determine the statistics we go back to the effective Lagrangian

$$\mathcal{L} = \frac{\kappa}{2}\left(\partial_0 \omega - \delta a_0\right)^2 - \frac{\rho_s}{2}\left(\vec{\nabla}\omega - \delta\vec{a}\right)^2$$
$$+ \frac{1}{2\pi}\varepsilon_{\mu\nu\lambda}\delta a^\mu \partial^\nu b^\lambda - \frac{k}{4\pi}\varepsilon_{\mu\nu\lambda} b^\mu \partial^\nu b^\lambda$$

and specialize for a field ω of a vortex.

~~The vortex has a vorticity~~

$$\frac{1}{2\pi}\varepsilon_{\mu\nu\lambda}\partial^\nu \delta a^\lambda = \Omega_\mu \quad \text{vorticity current}$$

$$\Rightarrow \mathcal{L} = \Omega_\mu b^\mu - \frac{k}{4\pi}\varepsilon_{\mu\nu\lambda} b^\mu \partial^\nu b^\lambda$$

$\Rightarrow$ vortex has statistics $\pm\frac{\pi}{k}$

<u>Vacuum Degeneracy</u> on a Torus can be read off this Lagrangian to be equal to $\underline{\underline{k}}$.

<u>How many $\neq$ vortices?</u>

The fundamental vortex has charge $\frac{e}{k}$ and statistics $\frac{\pi}{k}$. This is the Laughlin quasihole.

The quasiparticle currents are Wilson lines $e^{i\int_P a_\mu dx^\mu}$

Let's ~~set~~ call this operator $W_1[\Gamma]$.

Suppose that we have two Laughlin qp's

$$\Rightarrow e^{i\int_{\Gamma_1} dx_\mu a_\mu} e^{i\int_{\Gamma_2} dx_\mu a_\mu} \sim e^{i2\int_\Gamma dx_\mu a_\mu}$$

(as $\Gamma_1 \to \Gamma_2$: "fusion")

has a-charge of 2. The electric charge is $\frac{2}{k}e$ and the statistics is $\pm 4\frac{\pi}{k}$ ($= 2^2$)

We call this op. $W_2[\Gamma]$

If we fuse n qp's $\Rightarrow W_n[\Gamma]$ with charge $\frac{n}{k}e$ and statistics $\frac{n^2\pi}{k}$

However if $n = k \Rightarrow$ we get a state with charge e and statistics $k\pi$. Since $\underline{k}$ is <u>odd</u> this is an <u>electron</u> and a <u>fermion</u>.

$\Rightarrow$ This theory has k types of excitations which is <u>equal</u> to the degeneracy on a torus.

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We can now include the external probe e.m. field $\tilde{A}_\mu$

$$\mathcal{L} = \frac{K}{2}(\partial_0 \omega - \delta a_0 - e\tilde{A}_0)^2 - \frac{\rho_s}{2}(\vec{\nabla}\omega - \delta\vec{a} - e\vec{\tilde{A}})^2$$

$$+ \frac{1}{2\pi}\varepsilon_{\mu\nu\lambda}\,\delta a^\mu \partial^\nu b^\lambda - \frac{k}{4\pi}\varepsilon_{\mu\nu\lambda}\, b^\mu \partial^\nu b^\lambda$$

upon shifting $\delta a_\mu + e\tilde{A}_\mu \equiv \delta a_\mu$

$$\mathcal{L} = \frac{K}{2}(\partial_0 \omega - \delta a_0)^2 - \frac{\rho_s}{2}(\vec{\nabla}\omega - \delta\vec{a})^2$$

$$+ \frac{1}{2\pi}\varepsilon_{\mu\nu\lambda}\,\delta a^\mu \partial^\nu b^\lambda - \frac{e}{2\pi}\varepsilon_{\mu\nu\lambda}\,\tilde{A}^\mu \partial^\nu b^\lambda$$

$$- \frac{k}{4\pi}\varepsilon_{\mu\nu\lambda}\, b^\mu \partial^\nu b^\lambda$$

For vortexes we set $\Omega_\mu = \frac{1}{2\pi}\varepsilon_{\mu\nu\lambda}\partial^\nu \delta a^\lambda$

$$\boxed{\Rightarrow \mathcal{L} = \Omega_\mu b^\mu - \frac{e}{2\pi}\tilde{A}^\mu \varepsilon_{\mu\nu\lambda}\partial^\nu b^\lambda - \frac{k}{4\pi}\varepsilon_{\mu\nu\lambda} b^\mu \partial^\nu b^\lambda}$$

Clearly $\frac{\delta \mathcal{L}}{\delta \tilde{A}_\mu} = j_\mu = -\frac{e}{2\pi}\varepsilon_{\mu\nu\lambda}\partial^\mu b^\lambda$ (as in the hydro picture)

note: $\partial_\mu j^\mu = 0$