Lecture 27 12/1/2022

Composite Fermion Picture (Jain '89, López and EF '91)

Instead of mapping fermions $\longrightarrow$ bosons we will map fermions $\longrightarrow$ fermions $\Longrightarrow$ we will attach an <u>even</u> number (2s) of flux quanta to each electron $\Rightarrow$ ~~comp~~ composite fermion.

The action now is

$$S_F = \int d^3z \left\{ \psi^*(z)\,(i D_0 + \mu)\,\psi(z) + \frac{1}{2M} |\vec{D}\psi|^2 \right\}$$

$$- \frac{1}{2} \int d^3z \int d^3z' \left(|\psi(z)|^2 - \rho_0 \right) V(z-z') \left(|\psi(z')|^2 - \rho_0 \right)$$

$$+ \int d^3z \left[\frac{1}{2\pi} \epsilon_{\mu\nu\lambda}\, a^\mu \partial^\nu b^\lambda - \frac{(2s)}{4\pi} \epsilon_{\mu\nu\lambda}\, b^\mu \partial^\nu b^\lambda \right]$$

$$D_\mu = \partial_\mu + i\frac{e}{c} A_\mu + i a_\mu$$

$$V(z-z') = \delta(t-t')\, V(|\vec{z}-\vec{z}'|) \quad \text{(instantaneous)}$$

(except for issues of the global topology) we can integrate-out the b_μ field $\Rightarrow$ Chern-Simons term $\cancel{\left(\frac{4}{2s}\right) \frac{1}{4\pi} \epsilon_{\mu\nu\lambda} a^\mu \partial^\nu a^\lambda}$

$$\Rightarrow \mathcal{L}_{CS}(a) = \frac{1}{2s}\,\frac{1}{4\pi}\,\varepsilon_{\mu\nu\lambda}\, a^{\mu}\partial^{\nu}a^{\lambda}$$

Gauss Law $\Rightarrow$ $j_0(x) = \frac{-1}{2\pi(2s)}\,\varepsilon_{ij}\,\partial_i a_j$

$\Rightarrow$ average density $\rho_0 = \langle j_0(x)\rangle = \frac{N_e}{L^2}$

$\Rightarrow$ average flux $\mathcal{B}L^2 / \rho_0 = -\frac{1}{2\pi(2s)}\mathcal{B}$

$\mathcal{B}L^2 = 2\pi(2s)\,N_e$

$\Rightarrow$ total effective field $\frac{e}{c}B_{eff} = \frac{e}{c}B + \mathcal{B} \neq 0$

$N_\phi = BL^2 \Rightarrow \frac{e}{c}B_{eff}L^2 = \frac{e2\pi N_\phi}{c} - 2\pi(2s)\,N_e$

In the ground state $\langle \vec{j}\,\rangle = 0 \Rightarrow \langle \vec{\mathcal{E}}\,\rangle = 0$

Filling fraction $\nu = \frac{N_e}{N_\phi}$

$(e=c=1)$

$2\pi N_\phi^{eff} = 2\pi N_\phi - 2\pi(2s)\,N_e$

We have a problem of N_e electrons in N_ϕ^{eff} fluxes

$\Rightarrow$ fill up p effective Landau levels ($p \in \mathbb{Z}$)

$\Rightarrow \frac{N_e}{N_\phi^{eff}} = \pm p$, $\frac{N_e}{N_\phi} = \nu$ ($\ominus$ means

$\Rightarrow \pm\frac{1}{p} = \frac{1}{\nu} - (2s)$

$$\Rightarrow \quad \frac{1}{\nu} = \pm\frac{1}{p} + 2s \quad \Leftrightarrow \quad \nu = \frac{p}{2sp \pm 1}$$

These are the Jain fractions. If $p = +1$

$$\Rightarrow \text{ Laughlin states} \quad \nu = \frac{1}{2s+1} \equiv \frac{1}{m}$$

$$B_{eff} = B + \mathcal{B} = \cancel{\frac{B}{2sp \pm 1}} \quad \frac{B}{\pm 2sp + 1}$$

and a reduced eff. cyclotron frequency

$$\omega_c^{eff} = \frac{B}{|2sp \pm 1|}$$

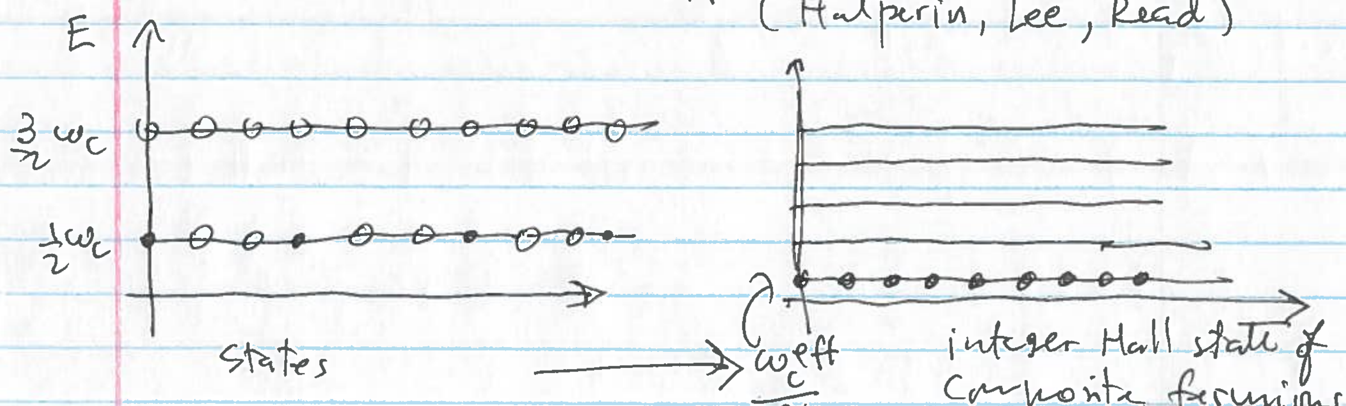
"Main series": $s = 1 \Rightarrow \nu = \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \frac{6}{13}, \ldots$

and

$$1, \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{5}{9}, \frac{6}{11}, \ldots$$

Notice that $\lim_{p \to \infty} \nu(s,p) = \frac{1}{2s}$ (even denominator)

but in this limit $B_{eff} \to 0$; Fermi Liquid? (Halperin, Lee, Read)



Compressible states

$\nu_\infty = \frac{1}{2s}$ and $B_{eff} = 0$

$\Rightarrow$ composite fermions fill up a Fermi sphere (at zero effective field)

$$\rho_0 = \frac{\nu_\infty}{2\pi \ell_0^2}$$

$$\int_{|p| \leq p_F} \frac{d^2 p}{(2\pi)^2} = \rho_0 \Rightarrow p_F = \frac{\sqrt{2\nu_\infty}}{\ell_0} \qquad (\hbar = 1)$$

$$\Rightarrow E_F = \frac{p_F^2}{2M} = \nu_\infty \frac{1}{M\ell_0^2}$$

Trial wave function: (for $\nu = 1/2$) (Haldane-Rezayi)

$$\Psi_{FL}(z_1 \dots z_N) = \underbrace{P_{LLL}}_{\text{projection}} \left[\det\left(e^{i\vec{k}\cdot\vec{r}_i}\right) \prod_{i<j} (z_i - z_j)^2 \right]$$

This is a theory of composite fermions at finite density at zero field coupled to a dynamical Chern-Simons gauge field

Effects of Quantum Fluctuations

Using the Gauss Law constraint the interaction term of the action becomes a functional of the gauge field a_μ

$$S_{int}[a_\mu] = -\frac{1}{2}\int d^3z\, d^3z' \left(\frac{B(z)}{2\pi\, 2s} - \rho_0\right) V(z-z') \left(\frac{B(z')}{2\pi\, 2s} - \rho_0\right)$$

We can now integrate out the fermions

$$S_{eff}[a_\mu] = -i\,\mathrm{tr}\ln\left[iD_0 + \mu + \frac{1}{2M}\vec{D}^2\right]$$

$$+ \frac{1}{2s} S_{cs}(a_\mu - A_\mu) + S_{int}(a_\mu - A_\mu)$$

probe field

$$S_{cs}(a) = \int d^3x\, \frac{1}{4\pi}\, \varepsilon_{\mu\nu\lambda}\, a^\mu \partial^\nu a^\lambda$$

$\Rightarrow$ Saddle Point:

$$\frac{\delta S_{eff}}{\delta a_\mu} = 0 \quad \Rightarrow \quad \text{mean field eqns.}$$

we discussed above.

Now a_μ are the fluctuations. Expanding to quadratic order in a_μ

$$S^{(2)}_{\text{eff}}(a_\mu) = \frac{1}{2}\int d^3x \int d^3y \; a_\mu(x)\, \tilde{\Pi}^{\mu\nu}_F(x-y)\, a_\nu(y)$$

$$+ \frac{1}{2s} S_{cs}(a_\mu - A_\mu) + S_{int}(a_\mu - A_\mu)$$

probe field

$\tilde{\Pi}^{\mu\nu}_F$ is the polarization tensor of the composite fermions in p filled eff. Landau levels.

Gauge invariance $\Longrightarrow$ $\partial_\mu \tilde{\Pi}^{\mu\nu}_F(x-y) = 0$ (transverse)

$$\tilde{\Pi}^F_{00} = \vec{Q}^2\, \tilde{\Pi}^F_0(\omega, \vec{Q})$$

$$\tilde{\Pi}^F_{0j} = \omega \vec{Q}_j\, \tilde{\Pi}^F_0(\omega, \vec{Q}) + i\,\varepsilon_{jk}\, Q_k\, \tilde{\Pi}^F_1(\omega, \vec{Q})$$

$$\tilde{\Pi}^F_{j0} = \omega \vec{Q}_j\, \tilde{\Pi}^F_0(\omega, \vec{Q}) - i\,\varepsilon_{jk}\, Q_k\, \tilde{\Pi}^F_1(\omega, \vec{Q})$$

$$\tilde{\Pi}^F_{ij} = \omega^2 \delta_{ij}\, \tilde{\Pi}^F_0(\omega, \vec{Q}) - i\,\varepsilon_{ij}\,\omega\, \tilde{\Pi}^F_1(\omega, \vec{Q})$$

$$+ (\vec{Q}^2 \delta_{ij} - Q_i Q_j)\, \tilde{\Pi}^F_2(\omega, \vec{Q})$$

The kernels $\tilde{\Pi}^F_0$, $\tilde{\Pi}^F_1$ and $\tilde{\Pi}^F_2$ represent particle-hole fluctuations of p-filled effective Landau levels. $\tilde{\Pi}^F_0$ and $\tilde{\Pi}^F_1$ are associated with time-reversal <u>even</u> processes and $\tilde{\Pi}^F_1$ with time-reversal <u>odd</u> processes.

For a Jain state with p filled effective Landau levels ~~each~~ the kernels ~~is~~ are given by a series of terms each of which have poles at the particle-hole excitation energies $\omega_{mn} = (m-n)\,\omega_c^{eff}$ (with $m > p$ (particle) and $n < p$ (hole)).

Each term has a residue of the form of a power of $\vec{Q}^{\,2}$ and a Laguerre polynomial.

Low energy regime: $\omega \ll \omega_c^{eff}$, $\vec{Q}$ small

$$\Rightarrow \tilde{\Pi}_0^F(0,0) = \frac{1}{2\pi}\frac{pM}{B_{eff}} \equiv \varepsilon \quad \text{(eff. dielectric constant)}$$

$$\tilde{\Pi}_1^F(0,0) = \pm \frac{p}{2\pi} \equiv \sigma_{xy}^{(0)} \quad \text{(Hall conductivity of the composite fermions)}$$

$$\tilde{\Pi}_2^F(0,0) = -\frac{1}{2\pi}\frac{p^2}{M} \equiv -\chi \quad \text{(eff. permeability)}$$

To leading order in ~~the~~ ω, and $\vec{Q}^2$ we find a local eff. Lagrangian (or action)

$$S_{eff} = \int d^3z \left(\frac{\varepsilon}{2} \vec{\mathcal{E}}^2 - \frac{\chi}{2} \mathcal{B}^2 \right) +$$

$$+ \left\{ \left(\sigma_{xy}^{(0)} + \frac{1}{2s} \right) S_{cs}(a_\mu) + \frac{1}{2s} S_{cs}(A_\mu) \right.$$

$$- \frac{1}{2\pi(2s)} \int d^3z\, \varepsilon_{\mu\nu\lambda}\, a^\mu \partial^\nu A^\lambda + S_{int}(a_\mu - A_\mu)$$

To obtain the full response to the external (probe) field A_μ we need to integrate out the dynamical field a_μ. The effective action for the probe field has the usual form

$$S_{eff}(A_\mu) = \frac{1}{2} \int d^3x \int d^3y\, A_\mu(x)\, \Pi^{\mu\nu}(x-y)\, A_\nu(y) + \cdots$$

where $\Pi^{\mu\nu}(x-y)$ is the full polarization tensor. By gauge invariance it has a decomposition of the same form as $\Pi_F^{\mu\nu}$

By explicit calculation one finds (Lopez, EF)

$$\Pi_{00}(\omega, \vec{Q}) = -\frac{\rho_0}{M} \frac{\vec{Q}^2}{\omega^2 - \omega_c^2 + i\epsilon} + \ldots$$

$$\omega_c = \frac{eB}{Mc} \equiv \frac{B}{M} \quad \text{(required by Kohn's Thm.)}$$

It satisfies the f-sum rule

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, i\omega \, D_{00}^{Ret}(\omega, \vec{Q}) = \frac{\rho_0}{M} \vec{Q}^2$$

$$D_{\mu\nu}^{Ret}(x-y) = -i\,\theta(x_0 - y_0) \langle [J_\mu(x), J_\nu(y)] \rangle$$

$$D_{00}^{Ret}(\omega, \vec{Q}) = -\Pi_{00}^{Ret}(\omega, \vec{Q})$$

Hall Conductance: keep only the leading terms

$$\mathcal{L}_{eff}(a_\mu, A_\mu) = \left(\sigma_{xy}^{(0)} + \frac{1}{2\pi 2s}\right) \frac{1}{2} \epsilon_{\mu\nu\lambda} a^\mu \partial^\nu a^\lambda$$

$$+ \frac{1}{4\pi(2s)} \frac{1}{2} \epsilon_{\mu\nu\lambda} A^\mu \partial^\nu A^\lambda$$

$$- \frac{1}{2\pi(2s)} \frac{1}{2} \epsilon_{\mu\nu\lambda} a^\mu \partial^\nu a^\lambda$$

$$Z(A_\mu) = \int \mathcal{D}a_\mu \, e^{i S_{eff}(a, A)}$$

$$\Rightarrow \mathcal{L}_{eff}(A_\mu) = \frac{1}{2} \sigma_{xy} \epsilon_{\mu\nu\lambda} A^\mu \partial^\nu A^\lambda$$

with $\sigma_{xy} = \frac{1}{2\pi} \left(\frac{1}{\pm\frac{1}{p} + 2s} \right) = \frac{1}{2\pi} \frac{p}{2sp \pm 1}$ (Jain)

Fractional Charge and Statistics

This requires the computation of a (product of) gauge invariant propagators

$$\langle \psi^{+}(x)\, e^{i \int_{\Gamma(x,x')} dz_\mu (A^\mu + a^\mu)} \psi(x') \rangle$$

gauge-invariant but path-dependent.

One expresses this propagator á la Feynman as a sum over histories. The exp. value then is a sum over histories of a heavy charged particle with a weight of the form

$$\langle e^{i \oint_\gamma a_\mu dx^\mu} \rangle_{CS}$$

where γ is the oriented union of the history and the path Γ.

$\Rightarrow$ one obtains a Bohm-Aharonov effect with an eff. charge

$$\frac{q_{eff}}{e} = 1 - \frac{\nu}{1/2s} = \frac{1}{2sp \pm 1}$$

Hence the excitations have fractional charge $\frac{e}{2sp \pm 1}$ (Laughlin $\frac{e}{2s+1} = \frac{e}{m}$)

To compute the statistics we need to calculate the 4 point function (i.e. the exp. value of <u>two</u> path-ordered operators)

This computation is decomposed into a sum of a direct and an exchange term with the exchange being the half-braid.

$$W_{ex} \propto e^{i\delta}$$

$$\delta = \pi \left(\frac{2s(p-1)+1}{2sp+1} \right) \pmod{2\pi}$$

For Laughlin $p=1 \Rightarrow \delta = \frac{\pm\pi}{m}$ $(m = 2s+1)$