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Quantum Hall Wavefunctions and CFT

Let's take a different look at the Laughlin wavefunction

$$\Psi_m(z_1,\dots,z_N) = \prod_{i<j}(z_i - z_j)^m \, e^{-\sum_{j=1}^{N} \frac{|z_j|^2}{4l_0^2}}$$

This expression has a striking resemblance to Coulomb gas results. To make this connection precise we will consider a chiral compactified scalar $\varphi(\vec{x})$ in 2D Euclidean space. The correlator of the compact scalar is $(z = x+iy)$

$$\langle \varphi(z)\, \varphi(z') \rangle = -\ln(z-z')$$

Note $\ln z = \ln|z| + i\arg z$

Compactness means that $\varphi(z)$ is periodic with R being the compactification radius

We will require that $\varphi(z) \simeq \varphi(z) + 2\pi n R$ with $n \in \mathbb{Z}$ $\Rightarrow$ the allowed observables are $V_p(z) = e^{ip\varphi(z)}$ s.t. $2\pi R p = 2\pi n$
(vertex operator)

$\Rightarrow$ the allowed values of $p = \dfrac{n}{R}$

Then the Laughlin wave function can be written as an expectation value of a set of vertex operators whose correlators are such that they describe fermions (electrons)

G. Moore and N. Read showed that

$$\Psi_m(z_1 \cdots z_N) = \left\langle \left(\prod_{i=1}^{N} e^{i\sqrt{m}\,\varphi(z_i)} \right) e^{-\int d^2z'\,\sqrt{m}\,\rho_0\,\varphi(z')} \right\rangle$$

chiral boson

with $R = \dfrac{1}{\sqrt{m}}$ and $\rho_0 = \dfrac{1}{2\pi m}$ (for $\nu = 1/m$)

We identify $e^{i\sqrt{m}\,\varphi(z)}$ as the operator that creates an electron ($\nu = 1/m$).

Quasihole: $V_{qh} = e^{\frac{i}{\sqrt{m}}\varphi(z)}$

$e^{-\frac{1}{4m}|z_0|^2}$

$$\Rightarrow \Psi(z_0; z_1, \ldots, z_N) = \prod_{j=1}^{N} (z - z_0)\, \Psi_m(z_1 \ldots z_N) \times$$

$$= \left\langle e^{\frac{i}{\sqrt{m}}\varphi(z_0)} \prod_{j=1}^{N} e^{i\sqrt{m}\,\varphi(z_j)}\, e^{-\int d^2z'\,\sqrt{m}\,\rho_0\,\varphi(z')} \right\rangle$$

This approach works for many other interesting states. For example a FQH fluid of a double well (a bilayer) has a wavefunction (Halperin)

$$\Psi_{m,m,n}(\{z_i\},\{w_i\}) =$$

$$= \prod_{i<j}(z_i-z_j)^m \prod_{i<j}(w_i-w_j)^m \prod_{i,j}(z_i-w_j)^n \times$$

$$\times\, e^{-\sum_{i=1}^{N}\frac{|z_i|^2+|w_i|^2}{4\ell_0^2}}$$

$\nu = 2p/(n+2s)p+1$ (Jain); $\nu = \dfrac{2}{n+m}$

Special case: $m=n+1 \Rightarrow$ z's are $\uparrow$ electrons
w's are $\downarrow$ electrons

This is a spin singlet state (unpolarized!)

$\swarrow (3,3,2)$

Singlet $\nu = \dfrac{2}{2n+1} = 2, \dfrac{2}{3}, \dfrac{2}{5}, \dfrac{2}{7} \ldots$

Another case of interest $(3,3,1)$ $(m=n+1=3)$

Spin-singlet states can be factorized into a spin singlet

$$\Psi_{singlet}(\{z_i^\uparrow\},\{z_i^\downarrow\}) = \prod_{i<j}\frac{(z_i^\uparrow-z_j^\uparrow)^{1/2}(z_i^\downarrow-z_j^\downarrow)^{1/2}}{(z_i^\uparrow-z_j^\downarrow)^{1/2}}$$

and a FQH state of semions:

$$\Psi_m^{semion}(\{z_i^\uparrow\},\{z_j^\downarrow\}) = \prod_{i<j}(z_i^\uparrow-z_j^\uparrow)^{n+\frac{1}{2}}(z_i^\downarrow-z_j^\downarrow)^{n+\frac{1}{2}}$$

$$\times (z_i^\uparrow-z_j^\downarrow)^{n+1/2} \times \exp(\ )$$

The semion wavefunction can be computed using chiral bosons

$$\Psi^{\text{semion}}_m([z_j^\uparrow],[z_j^\downarrow]) = \left\langle \prod_{i=1}^{N} e^{i\sqrt{n+\frac{1}{2}}\,\varphi(z_i)}\, e^{-\int d^2z' \sqrt{n+\frac{1}{2}}\,\rho_0\, \varphi(z')} \right\rangle_{U_b(1)}$$

The singlet wave function is an exp. value of vertex ops. in the $SU(2)_1$ CFT (WZW)

$$\Psi_{\text{singlet}}([z_i^\uparrow],[z_i^\downarrow]) =$$

$$= \left\langle V_{1/2}(z_1^\uparrow)\cdots V_{1/2}(z_{N/2}^\uparrow)\, V_{-\frac{1}{2}}(z_1^\downarrow)\cdots V_{-\frac{1}{2}}(z_{N/2}^\downarrow) \right\rangle_{SU(2)_1}$$

$V_{1/2}$ is the $j=1/2$ rep. of $SU(2)_1$

$SU(2)_1$ is <u>abelian</u> since $[1/2]\times[1/2]=[0]$

fusion

there's no $j=1$!

$V_{\pm 1/2}$ can also be represented in terms of a compactified boson $\phi(z)$ s.t.

$$V_{\pm 1/2}(z) = e^{\pm\frac{i}{\sqrt{2}}\phi(z)}$$

with compactification radius $R=\sqrt{2}$ (known as the $SU(2)$ radius)

In fact $J^3_R(z) \sim i\partial_z\phi$, $J^\pm_R(z)\sim e^{i\sqrt{2}\phi}$ are the $SU(2)_1$ currents

Moore-Read States and non-abelions

G. Moore and N. Read proposed the following wave function for the observed FQH plateau at $5/2$

$$\Psi_{MR}(z_1 \dots z_N) = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j)^n \, e^{-\sum_{j=1}^{N} \frac{|z_j|^2}{4\ell_0^2}}$$

$$\mathrm{Pf}(M_{ij}) = \frac{1}{2^{N/2}\left(\frac{N}{2}\right)!} \sum_{P} \mathrm{sgn}(P) \prod_{r=1}^{N/2} M_{P(2r-1),P(2r)}$$

antisymmetric matrix

permutations

$$\equiv \mathcal{A}(M_{12} \dots M_{N-1,N})$$

antisymmetrization

$$\mathrm{Pf}(M) = \pm\sqrt{\det M}$$

The filling fraction for this state is $\nu = \frac{1}{n}$

n even $\Rightarrow$ Ψ is antisymmetric $\Rightarrow$ fermions
n odd $\Rightarrow$ Ψ is symmetric $\Rightarrow$ bosons,

fermions $\nu = \frac{1}{2}, \frac{1}{4}, \dots$ even denominators!

The $n=2$ state ($\nu = 1/2$) is closely related to the Halperin (331) state (also with $\nu = 1/2$)

<u>CFT picture of the Moore-Read states</u>

$$\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) = \langle \chi(z_1) \cdots \chi(z_N) \rangle$$

$\chi(z)$ is the ~~MF~~ Majorana fermion (field)

of the (chiral) Ising CFT (free fermion!)

$$\langle \chi(z)\, \chi(u) \rangle = \frac{1}{z-u}$$

neutral sector

$$\Rightarrow \Psi_{MR}([z_i]) = \langle \chi(z_1) \cdots \chi(z_N) \rangle \times$$

$$\times \left\langle \prod_{i=1}^{N} e^{i\sqrt{n}\,\phi(z_i)}\, e^{-\int d^2z' \sqrt{n}\, \rho_0\, \phi(z_i)} \right\rangle_{U(1)_n}$$

charge sector

This is the exact G.S. wave function of

$$H = \sum_{i,\, j \neq i,\, k \neq i, j} P_{LLL}\, \delta'(z_i - z_j)\, \delta'(z_i - z_k)\, P_{LLL}$$

Ising CFT has three primaries $\mathbb{1}$, ~~ϵ~~ σ, χ

σ: Ising field (twist)

Fusion Rules: $\chi * \chi = \mathbb{1}$

$\sigma * \sigma = \mathbb{1} + \chi$ ⟵ two channels!

$\chi * \sigma = \chi$

The electron is $\chi(z)\, e^{i\sqrt{2}\,\phi(z)}$

The allowed states are created by operators that braid trivially with the electron:

They are: $\mathbb{1}$, $\underbrace{\sigma(z)\, e^{\frac{i}{2\sqrt{2}}\phi(z)}}_{\text{non-abelian}}$, $\underset{\text{Majorana}}{\chi(z)}$, $\underset{\text{Laughlin}}{e^{\frac{i}{\sqrt{2}}\phi(z)}}$

TQFT picture of the FQH states

This is the hydrodynamic theory!

$$\mathcal{L}[b_\mu] = -\frac{m}{4\pi}\, \varepsilon^{\mu\nu\lambda}\, b_\mu \partial_\nu b_\lambda - \frac{e}{2\pi} A^\mu \varepsilon_{\mu\nu\lambda}\, \partial^\nu b^\lambda$$

$$\underset{\text{vortex current}}{\pm\, j^\nu_\mu}\, b^\mu$$

$$\sigma_{xy} = \frac{1}{m}\frac{e^2}{h}, \quad \underset{\text{charge}}{Q = \pm\frac{e}{m}}, \quad \underset{\text{statistics}}{\delta = \frac{\pi}{m}}$$

Fractional spin S/ $2\pi S = \delta = \frac{\pi}{m} \Rightarrow S = \frac{1}{2m}$

<u>Coupling to the geometry</u>; spin connection ω_μ

Local frame (vielbeins) $e^a_j(t)$

$$\omega_0 = -\frac{\varepsilon^{ab}}{2} e^{aj} \partial_t e^b_j; \quad \omega_i = -\frac{\varepsilon^{ab}}{2} e^{aj} \partial_i e^b_j + \frac{\varepsilon^{jk}}{2\sqrt{g}} \partial_j g_{ik}$$

Wen-Zee term

$$\mathcal{L}_{WZ} = \frac{s}{2\pi}\,\varepsilon_{\mu\nu\lambda}\,\omega^{\mu}\partial^{\nu}b^{\lambda}$$

Shift: $N_\phi = \frac{1}{\nu} N_e - \mathcal{S}$ ← shift (electrons feel curvature as flux)

$\mathcal{S} \neq 0$ on a sphere (curved!)
$\mathcal{S} = 0$ on a torus (flat)

$$N_R = \frac{1}{2\pi}\int d\omega = \frac{1}{2\pi}\int R$$

Gauss-Bonnet: $N_R = 2(1-g)$ $\quad g$ = genus

Wen-Zee formula: $\mathcal{S} = \frac{\xi\, N_R}{2m} \equiv s\, N_R$

shift ($\mathcal{S}$) $\quad$ orb. spin (s)

$\Rightarrow$ sphere $N_R = 2 \Rightarrow$ Shift $\mathcal{S} = \frac{1}{m} = \nu$

torus $N_R = 0 \Rightarrow$ Shift $= 0$

Pretzel $N_R = -2 \Rightarrow$ Shift $\mathcal{S} = -\frac{1}{m}$

Effective TQFT For Multicomponent States

A multicomponent state with N components is described by a CS theory ~~with~~ with N hydrodynamic fields b_μ^I $(I = 1, \ldots, N)$

$$\mathcal{L}[b_\mu^I, A_\mu] = -\frac{1}{4\pi} K_{IJ}\, \varepsilon_{\mu\nu\lambda}\, b_I^\mu\, \partial^\nu b_J^\lambda$$

$$-\frac{e}{2\pi} A^\mu t_I\, \varepsilon_{\mu\nu\lambda}\, \partial^\nu b_I^\lambda$$

$$+ j_\mu (b_I^\mu l_I) + \frac{1}{2\pi} s_I\, \varepsilon^{\mu\nu\lambda} \omega_\mu \partial_\nu b_\lambda^I$$

t_I = charge vector (tells us how each component couples to A_μ)

l_I = how many # vortices

K_{IJ} is an integer-valued symmetric matrix with K_{11} = odd for electrons.

Vacuum degeneracy = $|\det K|$

Filling Fraction = $\nu = t_I (K^{-1})_{IJ}\, t_J$

Fractional charge = $Q = e\, t_I (K^{-1})_{IJ}\, l_J$

Fractional Statistics = $\delta = \pi\, l_I (K^{-1})_{IJ}\, l_J$

Shift = $t_I (K^{-1})_{IJ}\, s_J\, \nu^{-1} N_R$

Example

Halperin states $(N=2)$ (m,m,n)

$$K = \begin{pmatrix} m & n \\ n & m \end{pmatrix} \qquad m \text{ odd}$$

$$\vec{t} = (1,1)$$

$$\nu = \frac{2}{m+n}$$

$$Q = \pm \frac{e}{m+n}$$

$$\delta = \pi \frac{m}{m^2 - n^2}$$

We can define $b^{\mu}_{\pm} = \frac{1}{\sqrt{2}}\left(b^1_{\mu} \pm b^2_{\mu}\right)$

$$\mathcal{L} = \left(\frac{m+n}{4\pi}\right) \varepsilon_{\mu\nu\lambda}\, b^{\mu}_{+} \partial^{\nu} b^{\lambda}_{+} - \sqrt{2}\frac{e}{2\pi} A_{\mu}\, \varepsilon^{\mu\nu\lambda} \partial_{\nu} b^{+}_{\lambda}$$

$$+ \left(\frac{m-n}{4\pi}\right) \varepsilon_{\mu\nu\lambda}\, b^{\mu}_{-} \partial^{\nu} b^{\lambda}_{-} +$$

$$+ \partial^{qp}_{\mu} \left[\frac{1}{\sqrt{2}} (\ell_1 + \ell_2)\, b^{+}_{\mu} + \frac{1}{\sqrt{2}} (\ell_1 - \ell_2)\, b^{-}_{\mu} \right]$$

<u>Note</u> b^{-}_{μ} is charge neutral

* The state (331) has $m=3$, $n=1$

$$K = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \quad \vec{t} = (1,1), \quad \text{degeneracy on a torus} = 8$$

filling fraction $\nu = \frac{1}{2}$, $Q = \pm \frac{e}{4}$, $\delta = \frac{3\pi}{8}$

* The singlet state (332) has $m=3$, $n=2$

$$\nu = \frac{2}{5}, \quad Q = \pm \frac{e}{5}, \quad \delta = \frac{3\pi}{5} \qquad K = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$$

* (112)

$$K = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \vec{t} = (1,1)$$

$$\nu = \frac{2}{3}, \quad Q = \pm \frac{1}{3}, \quad \delta = \frac{-\pi}{3}$$

The gauge fields b_μ^I obey flux quantization conditions (compactness). The allowed charges (fluxes) form a hypercubic lattice. Redefinitions of the fields compatible with the charge lattice <u>describe the same state</u>

$$b_\mu^I \to W_{IJ} b_\mu^J \qquad \text{with } W \in SL(N, \mathbb{Z})$$

$$\Rightarrow t'_I = W_{IJ} t_J, \quad K'_{IJ} = W_{IK} K_{IL} W_{LJ}$$