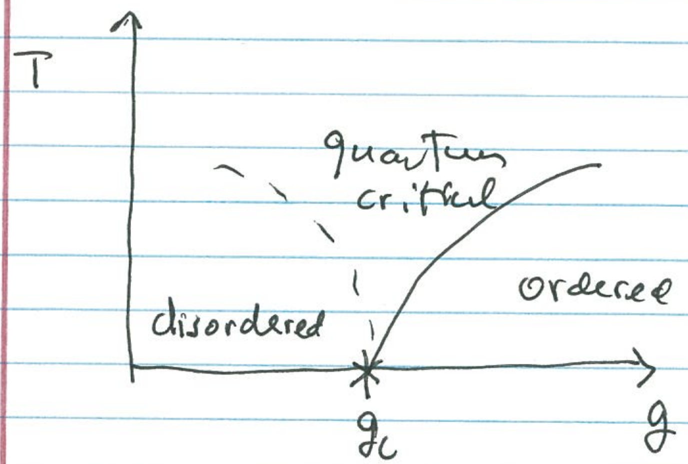
Phase Transition

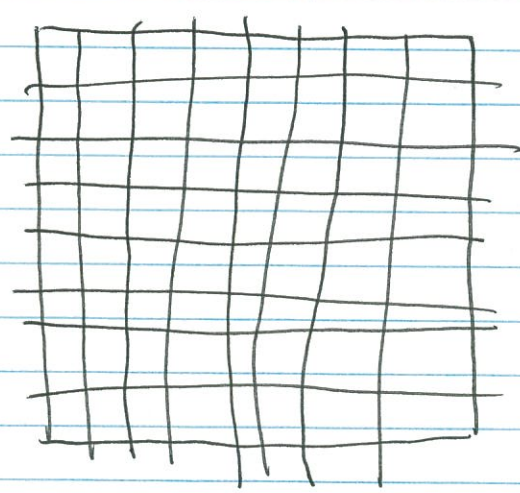


Classical - Quantum connection

We see that there is a similarity between quantum phase transitions and classical thermal transitions.

Why?

Examine the 2D Ising case



a 2d config. can be regarded as a sequence (evolution) of row configs.

row configs $\Longleftrightarrow$ 1d quantum states

⇒ the 2D configs are evolutions of row states ~~out~~ and the p.f. is the sum over such histories (evolutions)!

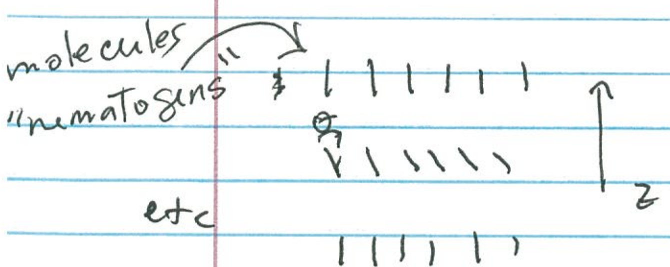
This is a path-integral in discretized (imaginary) time!

⇒ For simple Stat. Mech. problems which become isotropic at T_c ⇒ are equivalent to a relativistic quantum problem in one-dimension less.

(For a derivation see EF QFT Ch 14)

Is this always true?

No: For example a nematic liquid crystal ~~crystal~~



$$Z=\int \mathcal{D}\theta \; e^{-E[\theta]/T}$$

$$E[\theta]=\frac{1}{2}\int dz\int d^2x\,\left[(\partial_z\theta)^2+\kappa(\nabla^2\theta)^2\right]$$

more generally:

$$E(\theta)=\frac{1}{2}\int dz\int d^2x\left[(\partial_z\theta)^2+\rho(\vec{\nabla}\theta)^2+K(\nabla^2\theta)^2+\dots\right]$$

If $\rho > 0 \Rightarrow (\nabla^2\theta)^2$ is irrelevant ("nematic phase")

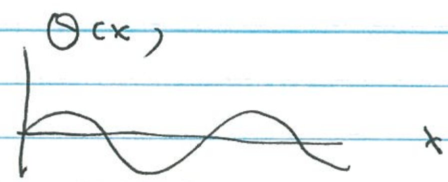
If $\rho < 0 \Rightarrow \theta(x)$ becomes modulated
("smectic phase")

why: in momentum space (in x-y plane)

$$-|\rho| k^2 |\theta(k)|^2 + \kappa k^4 |\theta(k)|^2$$

has a minimum at $\kappa |k_0|^2 = |\rho|$

$$\theta(x) = \tilde{\theta}(x) e^{i\vec{k}_0\cdot\vec{x}} + c.c$$



Transition from isotropic $\Rightarrow$ modulated

Lifshitz transition at $\rho = 0$

Quantum equivalent

$$H = \int d^2x \left(\frac{1}{2} \tilde{\Pi}^2 + \frac{\kappa}{2} (\nabla^2 \theta)^2 \right)$$

"Quantum Lifshitz"

dynamic crit. exp.

$$\omega(k) = \# |k|^2 \Rightarrow z = 2$$

General Picture (Landau)

* Global symmetry that is spontaneously broken. (SSB)
* SSB states are labelled by ~~a~~ a local order parameter $\phi(x)$ (~~observe~~ observables that transform under the symmetry)

In fact Landau (and Ginzburg) proposed a simple phenomenological theory. They postulated that the free energy has the form (D dimension)

$$F = \int d^Dx \left[\frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2(x) + \frac{u}{4}\phi^4(x) + \cdots \right]$$

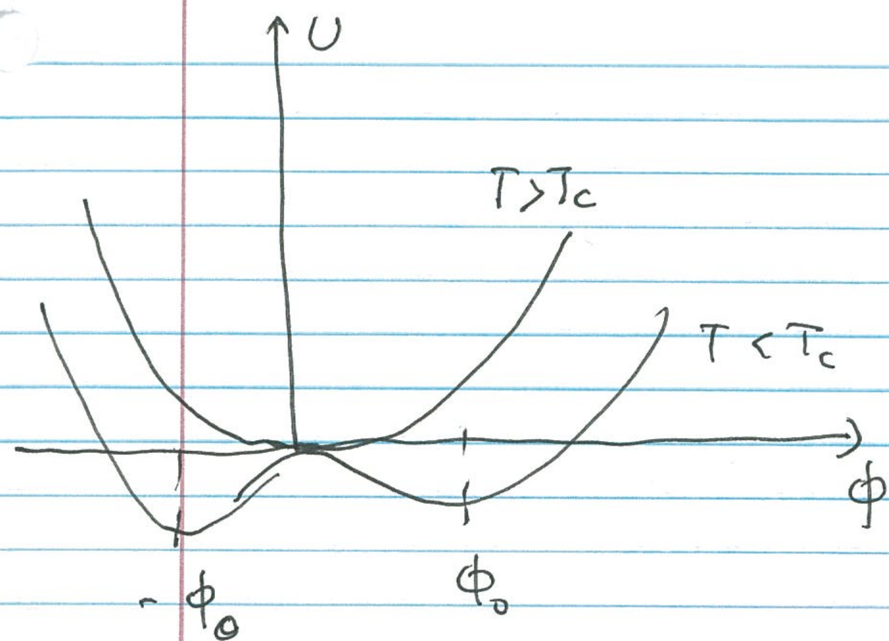
$$m^2 = a(T - T_c) \quad , \quad u > 0$$

(set the stiffness to 1)

$T > T_c \Rightarrow F$ has a minimum at $\phi = 0$

$T < T_c \Rightarrow F$ has a minimum at $|\phi| = \pm\phi_0$

$$\phi_0 = \sqrt{\frac{2a}{u}(T_c - T)}$$



This is a mean-field theory.

In general

$$Z = \int \mathcal{D}\phi \; e^{-S(\phi)}$$

path integrals in D Euclidian space-time.

$$S(\phi) = \int d^D x \left[\frac{1}{2} (\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2 + \frac{u}{4}\phi^4 \right]$$

choose x_D to be imaginary time

$$x_D \to i x_0 \quad \leftarrow \text{time}$$

$$-S(\phi) \to i S(\phi)$$

where now

$$S(\phi) = \int d^D x \left[\frac{1}{2} (\partial_0 \phi)^2 - \frac{1}{2} (\nabla \phi)^2 - \frac{m^2}{2} \phi^2 - \frac{u}{4} \phi^4 \right]$$

this is a relativistic scalar field in D dimensions!

Wilson developed the Renormalization Group to attack problems of this type

Close connection with the theory of phase transitions.

The problem in QFT is that perturbation theory in u is a series of divergent terms (Feynman diagrams)

RG solves this problem by tuning the theory to a fixed point associated with a phase transition where $a \ll \xi \ll L$

a: cutoff (lattice spacing)
L: size

$\Rightarrow$ $\xi \to \infty$ at the transition $\Longleftrightarrow$ continuum.

Quantum Systems at finite temperature

System with a Hamiltonian H and an observable ϕ (operators)

Thermal exp. value

$$\langle \phi \rangle = \frac{1}{Z} \operatorname{tr}\left(\phi\, e^{-\beta H}\right) \quad ; \quad H|n\rangle = E_n |n\rangle$$

$$Z = \operatorname{tr} e^{-\beta H} = \sum_n e^{-\beta E_n} \quad ; \quad \beta = \frac{1}{T}$$

$$\langle \phi \rangle = \frac{\sum_n \langle n | \phi | n \rangle\, e^{-\beta E_n}}{\sum_n e^{-\beta E_n}}$$

Def: $\phi(\tau) = e^{-\tau H}\, \phi\, e^{+\tau H}$

$$\partial_\tau \phi(\tau) = [\phi(\tau), H]$$

same as with q. evolution

$$\partial_t \phi(t) = i[\phi(t), H] \quad \text{(Heisenberg)}$$

$$\phi(t) = e^{-itH}\, \phi\, e^{itH}$$

Since we compute a trace $\Rightarrow$ the state at $\tau = 0$ and at $\tau = \beta = \frac{1}{T}$ must be the same (and arbitrary)

$\Rightarrow \phi(\tau) = \phi(\tau + \beta)$ periodic in imaginary time!

can expand in Fourier modes in τ

$$\Rightarrow \quad \phi(\tau) = \sum_{n=-\infty}^{+\infty} e^{i\omega_n \tau} \phi(\omega_n)$$

periodicity $\Rightarrow \quad \omega_n \beta = 2\pi n \Leftrightarrow \omega_n = \frac{2\pi n}{\beta} = 2\pi T n$

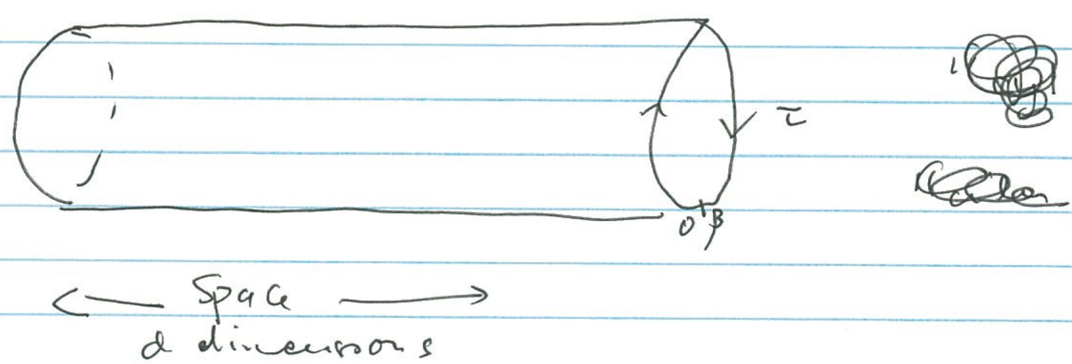
(Matsubara frequencies)

$\Rightarrow$ A quantum system at $T > 0$ looks like a classical system on a circle with circumference β!

$\Rightarrow$ In the case of a quantum phase transition with order parameter $\phi(x,t)$

$\Rightarrow \quad \phi(x,t) \to \phi(x,\tau) \qquad (t \to i\tau)$ "Wick rotation"

and compactify the (imaginary) time axis



If $\xi \gg \beta$ ("thermal wavelength")

we have a classical problem in $\underline{d}$ dimensions!

Why?

$$S(\phi) = \int_0^\beta d\tau \int d^dx \left(\frac{1}{2}(\partial_\tau \phi)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{u}{4}\phi^4 \right)$$

Expanding in modes

looks like a mass term!

$$S(\phi) = \frac{1}{\beta}\sum_n \int d^dx \Big[\frac{1}{2}\omega_n^2 |\phi(\omega_n, x)|^2 +$$

$$+ \frac{1}{2}\left(\nabla\phi(\omega_n, x)\cdot\nabla\phi^*(\omega_n, x)\right)$$

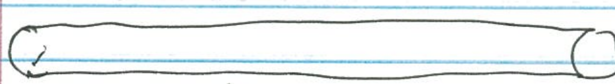
$$+ \frac{m^2}{2}|\phi(\omega_n, x)|^2$$

$$+ \text{same with } \phi^4 \Big]$$

If $n \neq 0 \Rightarrow \phi(\omega_n, x)$ has a mass $m^2 + \omega_n^2$

$(m = \frac{1}{\xi})$. If $\xi \gg \beta \Rightarrow \omega_n \gg \frac{1}{\xi} = m$

$\Rightarrow$ ~~as~~ all modes become "heavy" and only the $\omega_n = 0$ mode survives



$\xi \gg \beta$

$$\phi(0, x) = \phi^*(0, x)$$

$$S(\phi) = \int d^dx \left(\frac{1}{2}(\nabla\phi(0,x))^2 + \frac{m^2}{2}\phi(0,x)^2 + \frac{u}{4}\phi(0,x)^4 \right)$$

same as the classical theory in d dimensions.