Another example

Quantum antiferromagnets

$$H = J \sum_{\langle r, r' \rangle} \vec{S}(r) \cdot \vec{S}(r') \qquad J > 0$$

For $S = 1/2$, $\vec{S} = \frac{\vec{\sigma}}{2}$

$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

For spin $\underline{\underline{S}}$ one can use a semiclassical picture in terms of a polarization vector $\vec{n}$ which varies slowly if $S \gg 1$.

Non-linear sigma model (for a derivation see EF FT of CM ch 7)
(can be derived using coherent states)

Action $S = \int d^D x \; \frac{1}{2g} (\partial_\mu \vec{n})^2$

$g \sim \frac{1}{S}$ and $\underline{\underline{\vec{n}^2 = 1}}$ as a local constraint

$$(\partial_\mu \vec{n})^2 = \frac{1}{v_s} (\partial_0 \vec{n})^2 - v_s (\nabla_i \vec{n})^2$$

(relativistic
$v_s =$ spin wave

Here $\vec{n}$ describes the fluctuations around a classical Néel state

$$\vec{S}(x) = \vec{n}(x)\, e^{iQ\cdot x} + c.c.$$

$$= \vec{n}(x) \cos(\vec{Q}\cdot\vec{x})$$

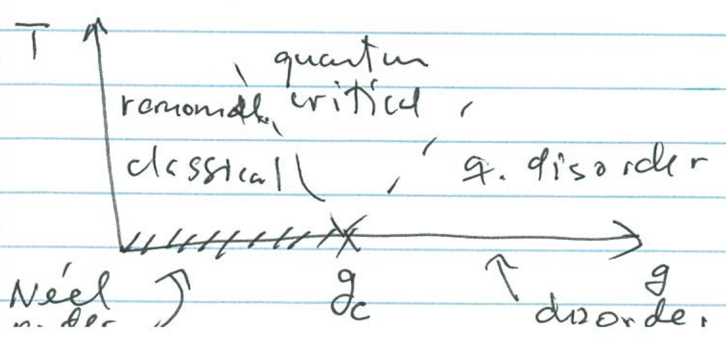
$$\vec{Q} = (\pi, \pi) \quad (\text{in } 2D)$$

This picture is correct in the phase with long range antiferromagnetic order (e.g La_2CuO_4)

The coupling constant $g \sim \frac{1}{S}$ can be modified by n.n.n. interactions J'

$$g' \simeq \frac{g}{\sqrt{1 - 2J'/J}} > g$$

If $J' \simeq J$ the NLSM has a phase transition to a disordered phase at same g_c



Two problems

(1) in $d=2$ $\quad J' \simeq J \Longrightarrow$ strong frustration

What is the nature of the q. disordered phase?

Is it a spin liquid?

(2) This picture is incorrect in $d=1$

Haldane $(d=1)$

$$Z = \int \mathcal{D}\vec{n} \; e^{-\frac{1}{2g}\int \underbrace{d^2x}_{dt\,dx} (\partial_\mu \vec{n})^2 + i\Theta Q[\vec{n}]}$$

(one can prove this using coherent states see EF 1 ch. 7)

where

$$Q[\vec{n}] = \frac{1}{8\pi}\int d^2x \; \varepsilon_{\mu\nu}\, \vec{n}(x)\cdot \partial_\mu \vec{n} \times \partial_\nu \vec{n}$$

is a topological invariant that counts the # of times $\vec{n}(x)$ sweeps the unit sphere $\vec{n}^2 = 1 \Longrightarrow$ it is an integer

The configs are homotopies with $\Pi_2(S_2) \simeq \mathbb{Z}$

Explicit calculation $\quad \Theta = 2\pi S$

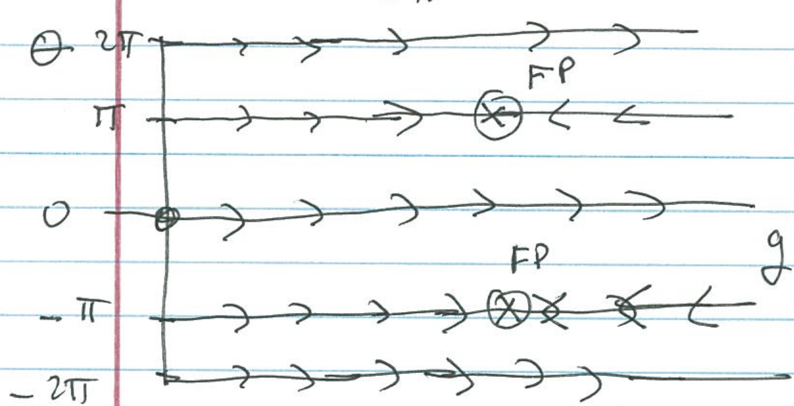
$$\Rightarrow e^{i\Theta Q[n]} = e^{i2\pi S\, Q[n]} = (-1)^{2S\,Q[n]}$$

$\Rightarrow$ if $S \in \mathbb{Z} \Rightarrow (-1)^{2SQ} = +1$

if $S = \frac{1}{2} + \mathbb{Z} \Rightarrow (-1)^{2SQ} = (-1)^{Q[n]}$

RG flow

$\beta(g) \simeq \frac{g^2}{2\pi} + \dots$ for small g (Polyakov 1975)



Topology matters!

(1) all spin S chains with S integer are gapped!

(2) all spin S chains with $S = \frac{1}{2} + \mathbb{Z}$ are gapless!!

Why? (1) The $S = 1/2$ chain is solvable by the Bethe Ansatz $\Rightarrow$ gapless (Explain)

(2) the FP for $S = \frac{1}{2} + \mathbb{Z}$ is a CFT (Wess-Zumino-Witten) ("$SU(2)_1$")

Q Are all phases of matter described by a local order parameter?

A No. We will see that other options exist $\Longleftrightarrow$ topological phases of matter

The simplest example is a $\mathbb{Z}_2$ lattice gauge theory. These theories were introduced in HEP by Wilson (and by Kogut and Susskind) to study the problem of quark confinement.

We will discuss two examples that are of interest in CMP.

1. $\mathbb{Z}_2$ gauge theory ("Ising Gauge Theory")
2. $U(1)$ gauge theory ("compact QED")

Maxwell's Electrodynamics

$$\mathcal{L} = \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right)$$

$$\left.\begin{aligned} \vec{E} &= -\partial_0 \vec{A} - \vec{\nabla} A_0 \\ \vec{B} &= \vec{\nabla} \times \vec{A} \end{aligned}\right\}$$ invariant under local gauge transf.

$A_\mu \to A_\mu + \partial_\mu \Phi$ (Φ smooth

* Quantization in the $A_0 = 0$ gauge

$$\vec{E} = -\partial_0 \vec{A} \Rightarrow \vec{E} = -\vec{\Pi}$$

canonical momentum conjugate to $\vec{A}$

(equal-time)

$$\Rightarrow [A_j(\vec{x}), E_k(\vec{y})] = -i\delta_{jk}\, \delta^3(\vec{x}-\vec{y})$$

$$H = \int d^3x \; \frac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right)$$

kinetic potential

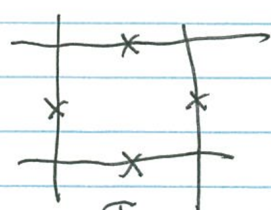
Constraint on the Hilbert space

Gauss's Law $\quad \vec{\nabla}\cdot\vec{E}\,|\text{Phys}\rangle = 0$

$\vec{\nabla}\cdot\vec{E}(x)$ is the generator of local time-indep. gauge transf.

$\mathbb{Z}_2$ gauge theory

Here → $\mathbb{Z}_2$ gauge fields defined on the links of a lattice. ① Euclidean, ② Hamiltonian.



σ_μ

① Euclidean version (D dimensions)

$$Z = \sum_{[\sigma_\mu]} e^{-S[\sigma]}, \quad S = -K \sum_{\text{plaquettes}} \sigma_\mu \sigma_\nu \sigma_\mu \sigma_\nu$$

$$K = \frac{1}{g}$$