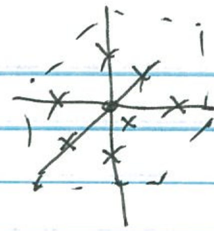
L5 9/8/2022

Gauge invariance

flip all fields

on links attached

to the same site $\underline{x}$



This is an analog of the Ising model but

with a local symmetry

The observables must be gauge-invariant

Wilson Loops

are invariant

$$W_\Gamma = \prod_{(x,\mu)\in\Gamma} \sigma_\mu(x)$$

Loops: worldlines of test particles

Γ: closed loop on the lattice

* Elitzur's Theorem: Gauge symmetries cannot be spontaneously broken.

Nevertheless, Wilson loops have two possible behaviors

(A) $K \ll 1$ $(g \gg 1)$ ("strong coupling")

(analog of high T in the Ising model)

$$\langle W_\Gamma \rangle \sim e^{-\sigma \, \mathrm{Area}(\Gamma)}$$

Area Law

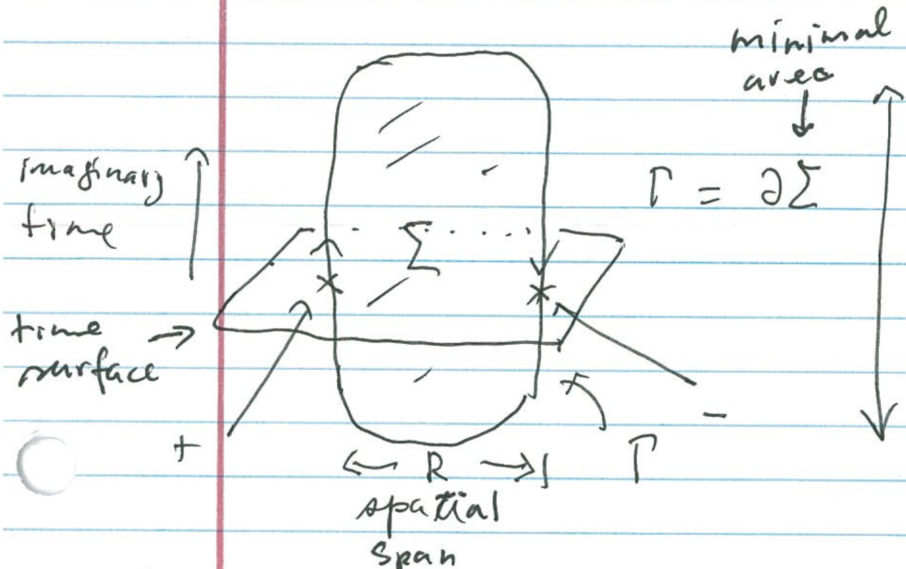
σ: "string tension"

(fastest rate of decay)

(B) $K \gg 1$ $(g \ll 1)$ (weak coupling)

$$\langle W_\Gamma \rangle \sim e^{-\rho L(\Gamma)} \qquad \text{Perimeter Law}$$

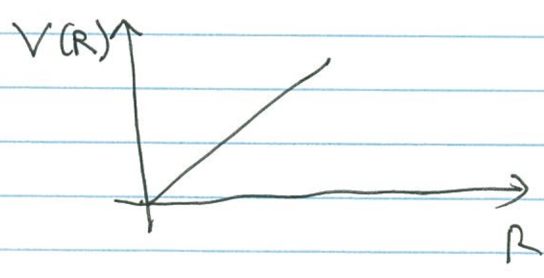
choose one direction as imaginary time



Effective potential $V(R) = \lim_{T \gg R} \left[-\frac{1}{T} \ln \langle W_\Gamma \rangle \right]$

(A) $V(R) = \lim_{T \gg R} \left(\frac{1}{T} \sigma \, \text{Area}(\Gamma) \right) = \sigma R$

(Area $= RT$)



confinement of static charges

B $$V(R) = \lim_{T \gg R} \rho\left(\frac{R+T}{T}\right) \simeq$$

$$= \rho \quad \text{independent of } R$$

$\Rightarrow$ sources are screened , ρ : 2 x self energy

(up to $e^{-R/\xi}$ corrections) (not confined!)

Hamiltonian Picture

$$H = -\sum_{(\vec{r},\mu)} \sigma^1_\mu(\vec{r}) - \frac{1}{g} \sum_{(\vec{r},\mu\nu)} \sigma^3_\mu(r)\sigma^3_\nu(r)\sigma^3_\mu(r)\sigma^3_\nu($$

links

Gauge - invariance:

$$Q(\vec{r}) = \prod_{(r,\mu)} \sigma^1_\mu(\vec{r})$$

links that ~~share~~ share the site $\vec{r}$

$$[H, Q(r)] = 0 = [Q(r), Q(r')]$$

locally

Constraint $\quad Q(\vec{r})|\text{Phys}\rangle = |\text{Phys}\rangle$

~~&~~ Two phases

(A) Strong coupling, $g \gg 1$

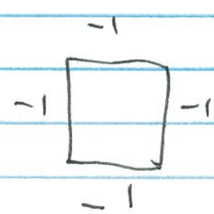
$H = H_0 + V$, $H_0 = -\sum\limits_{\underset{\uparrow \text{ links}}{(r,\mu)}} \sigma^1_\mu(r)$

$$V = -\frac{1}{g} \sum_{\text{plaquettes}} \sigma^3_\mu \sigma^3_\nu \sigma^3_\mu \sigma^3_\nu$$

"Vacuum" state & ~~$\sigma^1_\mu(r)\,|\text{Phys}\rangle = -\,|\text{Phys}\rangle$~~

$\sigma^1_\mu(r)\,|\Psi\rangle = +\,|\Psi\rangle$ (gauge invariant)

First excited state: small loop around an elementary plaquette



+1 elsewhere

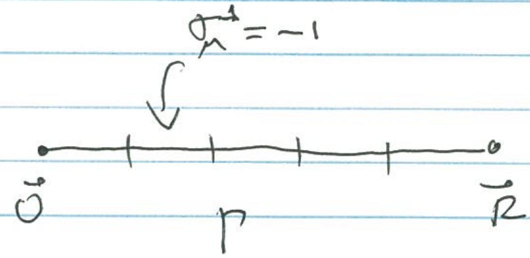
As g ~~goes~~ ~~to~~ decreases larger loops contribute
The loop basis is natural for $g \gg 1$.

(B) Confinement? Introduce 2 static sources: at $\vec{0}$ an $\vec{R}$

$Q(\vec{0}) = -1$ and $Q(\vec{R}) = -1$

$Q(\vec{r}) = +1$ elsewhere

Gnd state with two sources



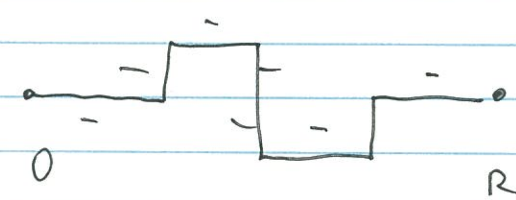
$\sigma^1_\mu(r) = -1$ on the shortest path

$\sigma^1_\mu(r) = +1$ elsewhere

Energy $= 2 \times L(\Gamma) = \underset{\uparrow}{2|R|}$

"string tension"

As g decreases the string fluctuates



also contribute (plaquette operators flip σ^1's on a loop)

B Weak coupling $\quad g \ll 1$

$$H_0 = -\frac{1}{g} \sum \sigma^3_\mu \sigma^3_\nu \sigma^3_\mu \sigma^3_\nu \qquad \text{with } Q = +1 \text{ every where}$$

two representations:

1 in the σ^3 basis: fix the gauge $\sigma^3_1 = +1$ (forexample)

the vac. state has $\sigma^3_\mu = +1$ every where

~~This is not a gauge-inv. state!~~ (need gauge fixing)

This is <u>not</u> a gauge-invariant state!

Alternative ~~rep~~.: use the σ^1 representation

$$|\Psi\rangle = \sum_{\text{loops}} |\text{loop}\rangle \qquad \text{(Kitaev)}$$

where $|\text{loop}\rangle$ is a state with arb. closed loops where $\sigma^1_\mu = -1$

This is a gauge-invariant state

$$Q(r)\,|\Psi\rangle = |\Psi\rangle$$

arbitrary

Sources? open loops with endpoints at $\vec{0}$, and $\vec{R}$

$\Rightarrow$ ~~the~~ Energy = self-energy

This phase can be viewed as a state in which the "electric" loops proliferate!

Q: Is the state $|\Psi\rangle$ unique?

(A) it depends on the boundary conditions

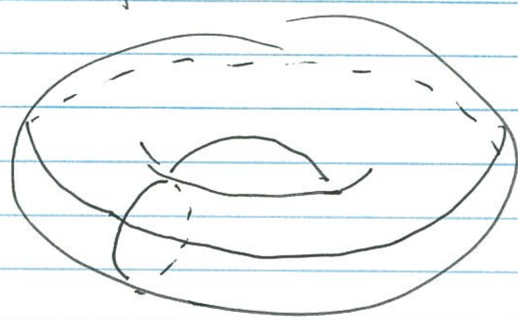
(1) fixed BC's: unique state ~~one~~

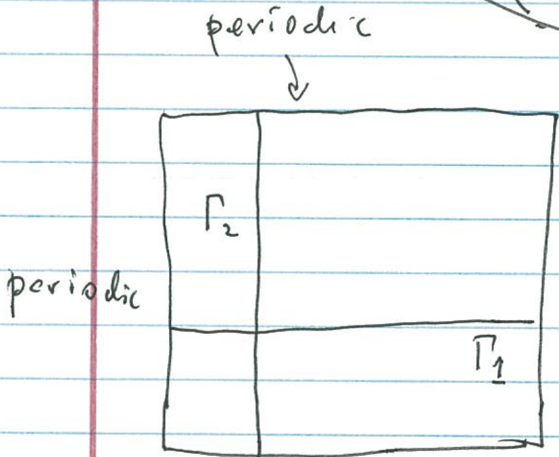
(2) periodic BC's: the state is degenerate

The degeneracy depends on the topology of space.

* Global Observables (d=2 as an example)

① Wilson loops on non-contractible cycles

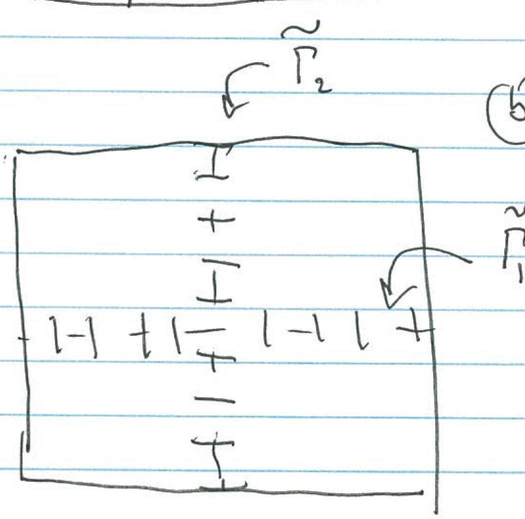




ⓐ "electric loops" (Wilson)

$$W[\Gamma_1] = \prod_{(r,\mu)\in\Gamma_1} \sigma^3_\mu(r)$$

$$W[\Gamma_2] = \prod_{(r,\mu)\in\Gamma_2} \sigma^3_\mu(r)$$



ⓑ "magnetic loops" ('t Hooft)

$$\tilde W[\tilde\Gamma_1] = \prod \sigma^1_\mu(r)$$

links crossed by $\tilde\Gamma_1$

same with $\tilde W[\tilde\Gamma_2]$

$$[W[\Gamma_1], W[\Gamma_2]] = 0$$

$$[\tilde W[\tilde\Gamma_1], \tilde W[\tilde\Gamma_2]] = 0$$

<u>but</u> $\{W[\Gamma_1], \tilde W[\tilde\Gamma_2]\} =$

$\{W[\Gamma_2], \tilde W[\tilde\Gamma_2]\} = 0$

$$[W[\Gamma_1], H_0] = [W[\Gamma_2], H_0] = 0 \qquad (g \to 0)$$

Also $[\tilde{W}[\tilde{\Gamma}_1], H_0] = [\tilde{W}[\tilde{\Gamma}_2], H_0] = 0$

and are gauge invariant

In addition:

$$W[\Gamma_1]^2 = I = W(\Gamma_2]^2$$

$$\tilde{W}[\tilde{\Gamma}_1]^2 = I = \tilde{W}[\tilde{\Gamma}_2]^2$$

If $|\Psi\rangle$ is an eigenstate of H_0

and of $W[\Gamma_1]$ with e.v. $+1$

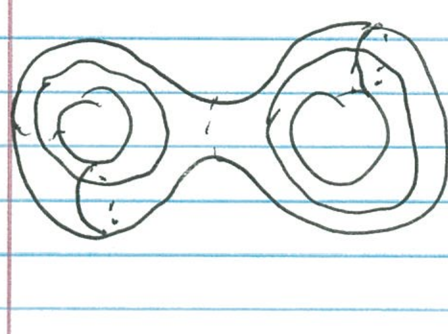
$\Rightarrow$ $\tilde{W}[\tilde{\Gamma}_2]|\Psi\rangle$ has e.v. -1

also $W(\Gamma_2]$ has e.v. $+1$

and $\tilde{W}[\tilde{\Gamma}_1]|\Psi\rangle$ has ev. -1

$\Rightarrow$ In $d=2$ we have 4 states

* 2 states for each non-contractible loop.



genus = # of handles

$g=0$ sphere

$g=1$ torus

$g=2$ pretzel

degeneracy $= 2^g$

$\Rightarrow$ the $g \to 0$ phase is topological

$\Rightarrow$ ground state degeneracy not associated with spontaneous symmetry breaking but with topology

Later we will see that these states have large scale entanglement!

These are properties of the entire phase $g < g_c$, not just of $g \to 0$