Gauge invariance

Flip all fields

on links attached to the same site $x$

This is an analog of the Ising model but with a local symmetry

The observables must be gauge-invariant

Wilson loops $W_L = \prod_{x \in L} \phi(x)$ are invariant

Loops: worldlines of test particles closed loop in the lattice

* Elitzur's Theorem: Gauge symmetries cannot be spontaneously broken.

Nevertheless, Wilson loops have two possible behaviors

1. $K \ll 1$ ($g \gg 1$) ("strong coupling") (analog of high $T$ in the Ising model)

   $\langle W_L \rangle \sim e^{-\frac{\text{Area}(L)}{K^2}}$ (fastest rate of decay)

   Area Law "string tension"
\( k \gg 1 \quad (g \ll 1) \quad \text{(weak coupling)} \)

\[ \langle W_p \rangle \sim e^{-\frac{\text{Area}(\Gamma)}{T}} \quad \text{Perimeter Law} \]

Choose one direction as imaginary time

[Diagram showing minimal area and time span]

\[ \Gamma = \text{min} \]

\[ T \text{ (time span) } \quad \text{not temperature!} \]

Effective potential \( V(R) \) as \( T \gg R \):

\[ V(R) = \lim_{T \gg R} \left[ -\frac{1}{T} \ln \langle W_p \rangle \right] \]

\[ (R) \quad V(R) = \lim_{T \gg R} \left( \frac{1}{T} \text{Area}(\Gamma) \right) = \sigma R \]

\[ \text{Area} = RT \]

Effective potential \( V(R) \) as \( T \gg R \):

\[ \text{confinement of static charges} \]

\[ R \]
\( V(\rho) = \lim_{T \to R} \frac{g(T + \frac{\pi}{2})}{T} \)

\( = g \text{ independent of } R \)

Sources are screened, \( g = 2 \times \text{self-energy} \)

(up to \( e^{-R/\rho} \text{ corrections} \) (not confirmed!))

Hamiltonian Picture

\[
H = -\sum_{\langle \mu, \nu \rangle} \frac{q_0}{\mu} \sigma^z(\tau) - \sum_{\langle \mu, \nu \rangle} \frac{q_1}{\mu} \sigma^z(\tau) \sigma^z(\tau') \sigma^z(\tau')
\]

links

Gauge invariance:

\[
Q(\tau) = \prod_{\langle \mu, \nu \rangle} \sigma^z(\tau) \mu
\]

Brackets that

\[ [H, Q(\tau)] = 0 = [Q(\tau), Q(\tau')] \]

locally

Constraint \( Q(\tau) |\text{Phys} > = |\text{Phys} > \)
Two phases

(a) Strong coupling, \( g \gg 1 \)

\[ H = H_0 + V , \quad H_0 = - \sum_{c(\mu)} \sigma^\mu(r) \]

\[ V = - \frac{1}{2} \beta \sum_{\text{plaquettes}} \sigma^3 \sigma^3 \sigma^3 \sigma^3 \]

"Vacuum" state \( \psi \rangle \) is gauge invariant

First excited state: small loop around an elementary plaquette

As \( g \) decreases, larger loops contribute

The loop basis is natural for \( g \gg 1 \)

(b) Confinement? Introduce 2 static sources: at \( \bar{0} \) and \( \bar{1} \)

\( Q(\bar{0}) = -1 \) and \( \bar{Q}(\bar{1}) = -1 \)

\( Q(\bar{1}) = +1 \) elsewhere
Ground state with two sources

\[ \sigma_3 = -1 \quad \sigma_3^\mu = -1 \quad \text{on the} \]

shortest path

\[ \sigma_3 \quad \sigma_3^\mu \quad \text{on a loop} \quad \sigma_3^\mu = +1 \quad \text{elsewhere} \]

Energy \[ E = 2 \times L(\sigma_3) = 2 \, |R| \]

"string tension"

As \( g \) decreases the string fluctuates

\[ \text{also contribute (plaquette operators flip outside a loop)} \]

**B** Weak coupling \( g \ll 1 \)

\[ H_0 = -\frac{i}{g} \sum_{\mu < \nu} \sigma_3^\mu \sigma_3^\nu \quad \text{with} \quad Q = +1 \quad \text{everywhere} \]

two representations:

1. in the \( \sigma_3 \) basis: fix the gauge \( \sigma_3^3 = +1 \) (for example)

the vac. state has \( \sigma_3^\mu = +1 \) everywhere

This is not a vac. state! (need gauge)

This is not a gauge-invariant state! fixing
Alternative : use the representation

\[ |\Psi\rangle = \sum |\text{loop}\rangle \quad \text{(Kitaev)} \]

where \(|\text{loop}\rangle\) is a state with arb. closed loops where

\[ \sigma^z = -1 \]

This is a gauge-invariant state

\[ \Theta(r) |\Psi\rangle = |\Psi\rangle \]

arbitrary

Sources: open loops with endpoints at \(0\) and \(R\)

\[ \Rightarrow \text{loop Energy = self-energy} \]

This phase can be viewed as a state in which the "electric" loops proliferate!

Q: Is the state \(|\Psi\rangle\) unique?

\(1\) it depends on the boundary conditions

\(1\) fixed BC's: unique state

\(2\) periodic BC's: the state is degenerate

The degeneracy depends on the topology of space.
* Global Observables (d=2 as an example)

1. Wilson loops on non-contractible cycles

![](torus.png)

2. "Electric loops" (Wilson)

\[ W[\Gamma_1] = \prod \delta_{\mu}(r) \delta_{\nu}(r) \exp i \pi \]

3. "Magnetic loops" ("t Hooft")

\[ \tilde{W}[\tilde{\Gamma}_1] = \prod \delta_{\mu}(r) \exp i \pi \]

\[ \tilde{W}[\tilde{\Gamma}_2] = 0 \]

[\[ W[\Gamma_1], W[\Gamma_2] \] = 0 but \[ \tilde{W}[\tilde{\Gamma}_1], \tilde{W}[\tilde{\Gamma}_2] \] = 0]

\[ [\tilde{W}[\tilde{\Gamma}_1], \tilde{W}[\tilde{\Gamma}_2]] = 0 \]
\[ [W[r_1], H_0] = [W[r_2], H_0] = 0 \quad (y \to 0) \]

Also \[ [\tilde{W}[\tilde{r}_1], H_0] = [\tilde{W}[\tilde{r}_2], H_0] = 0 \]

and are gauge invariant.

In addition:
\[ W[r_1]^2 = 1 = W[r_2]^2 \]
\[ \tilde{W}[\tilde{r}_1]^2 = 1 = \tilde{W}[\tilde{r}_2]^2 \]

If \( |\Phi\rangle \) is an eigenstate of \( H_0 \)
and if \( W[r_1] \) with e.v. \( +1 \)
\[ \Rightarrow \tilde{W}[\tilde{r}_2]|\Phi\rangle \] has e.v. \(-1\)

also \( W[r_2] \) has e.v. \(+1\)
and \( \tilde{W}[\tilde{r}_1]|\Phi\rangle \) has e.v. \(-1\)

0) In \( d = 2 \) we have 4 states
\* 2 states for each non-contractible loop.

\[ \text{genus} = \# \text{ of handles} \]
\[ g = 0 \text{ sphere} \]
\[ g = 1 \text{ torus} \]
\[ g = 2 \text{ pretzel} \]
the $g \to 0$ phase is topological

ground state degeneracy not associated with spontaneous symmetry breaking but with topology

Later we will see that these states have large scale entanglement!

There are properties of the entire phase $g < g_c$, not just of $g \to 0$