L6 9/8/2022

Ising Duality

Two pictures: (1) Kramers-Wannier-Wegner

(2) Hamiltonian picture

(1) ~~[illegible]~~

(A) Write the Partition Function as a sum over loops (high T) $(\beta = J/T)$

$$Z = \sum_{[\sigma]} e^{\beta \sum_{n.n} \sigma\sigma} =$$

on a link

$$= (\cosh\beta)^{N_L} \sum_{[\sigma]} \prod_{\text{links}} (1 + \tanh\beta\, \sigma\sigma)$$

N_L : # of links of the lattice

N_p = # of plaquettes

$$\sigma^2 = 1 \Rightarrow \frac{Z}{(\cosh\beta)^{N_L}} = 1 + 2\tanh^4\beta^{N_p} + \dots = \tilde{Z}$$

$\square$

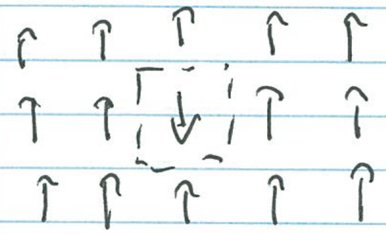
$\tilde{Z}$ = sum over closed (non-overlapping) loops $\{\Gamma\}$ each with a weight $(\tanh\beta)^{L(\Gamma)}$

* True in all dimensions
* looks like a path integral for heavy particles which interact at crossing points.

(B) (I) In 2D (low T)

Expand around the ordered state $\uparrow \cdots \uparrow$

First correction: domain walls



Sum over configs. of closed non-overlapping domain walls Γ (dual lattice)

weight $= e^{-2\beta L[\Gamma]}$

Kramers-Wannier: two expansions in loops
('41)

$\tanh\beta \longleftrightarrow e^{-2\beta}$

Phase transition at a self-dual point?

$\Rightarrow \quad e^{-2\beta_c} = \tanh\beta_c$

$\Rightarrow \quad \beta_c = \frac{J}{T_c} = \frac{1}{2}\ln(\sqrt{2}+1)$ (Onsager's T_c)
('44)

(II) 3D (Wegner) ('71)

Z is also a sum over closed domain walls but in 3D domain walls are closed surfaces!

$\Rightarrow$ The dual is a theory with closed surfaces (not the high T phase of an Ising Model)

The dual of the 3D Ising Model is the 3D Ising Gauge Theory! (Wegner)

Why?

$$Z = \sum_{[\sigma_\mu]} e^{\beta \sum_{\text{plaquettes}} \sigma\sigma\sigma\sigma}$$

* For $\beta \ll 1$ is a sum over closed surfaces Σ with a weight $(\tanh\beta)^{\text{Area}(\Sigma)}$

* For $\beta \gg 1$ expand around the config $\sigma = 1$ (everywhere)

$\Rightarrow$ Sum over closed loops Γ with a weight $e^{-2\beta L(\Gamma)}$

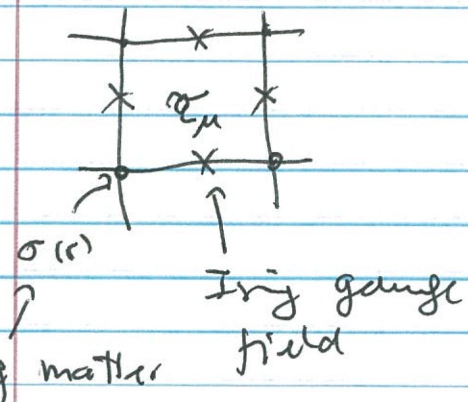
$$\Rightarrow \quad e^{-2\beta_{\text{Ising}}} = \tanh \beta_{\text{gauge}}$$

$$e^{-2\beta_{\text{gauge}}} = \tanh \beta_{\text{Ising}}$$

No self-duality

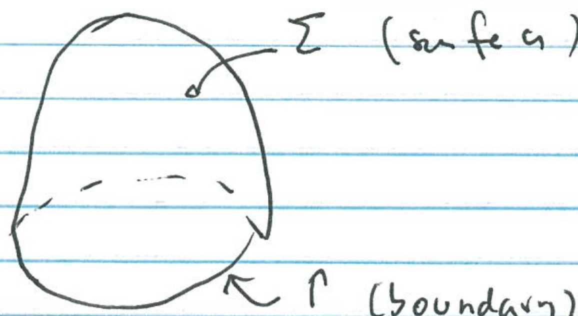
$\Rightarrow$ Confined phase (gauge theory) $\xleftrightarrow{\text{dual}}$ broken symmetry state of Ising Model

deconfined phase (topological) $\longleftrightarrow$ disordered phase of the Ising Model self-dual model: Ising gauge theory with Ising matter field

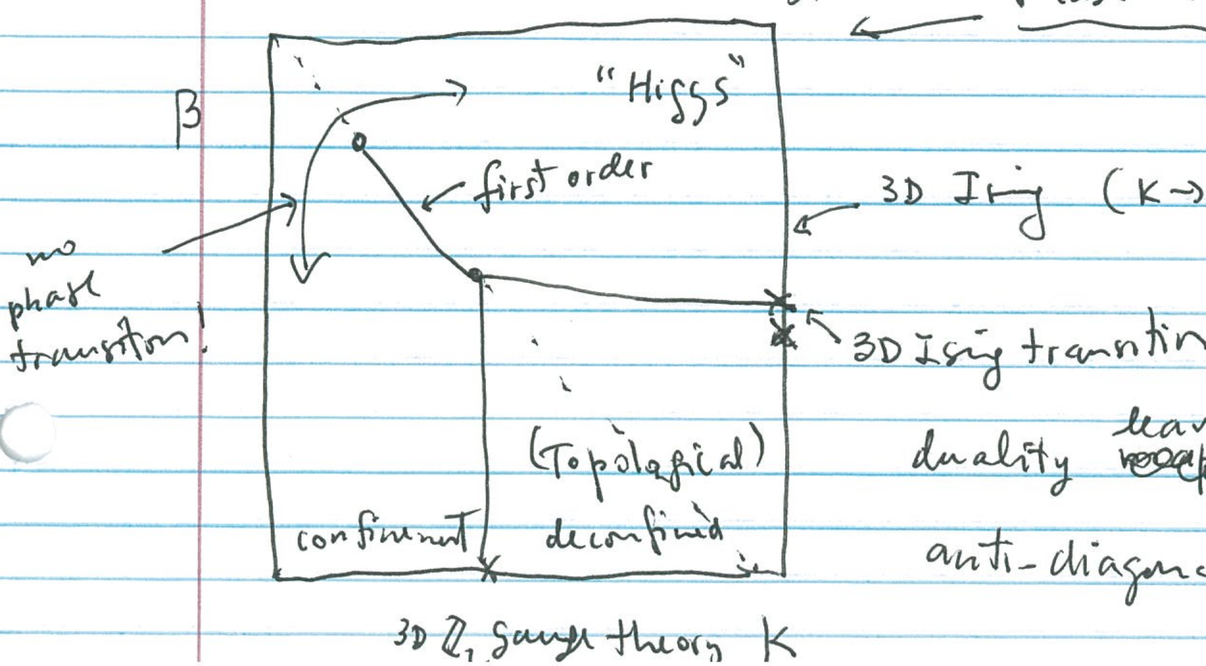


$$Z = \sum_{[\sigma],[\tau_\mu]} \exp\left(\beta \sum_{\text{links}} \sigma\tau\sigma + K \sum_{\text{plaquettes}} \tau\tau\tau\tau\right)$$

Z is a sum over closed surfaces and surfaces with boundaries (attached to matter fields) "loops"



Phase diagram



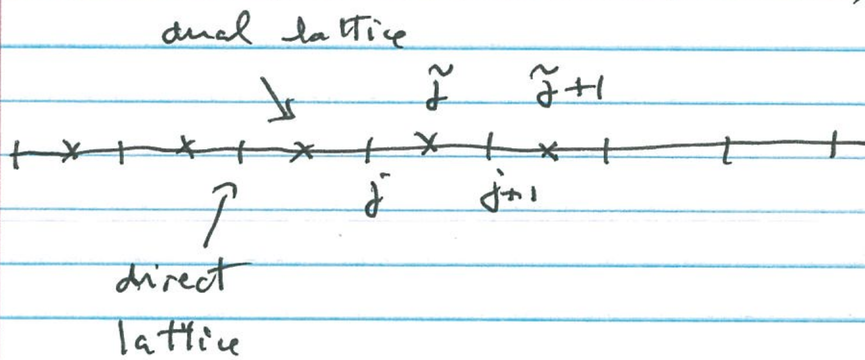
duality ~~maps the~~ leaves the anti-diagonal invariant

Hamiltonian Duality (Fradkin – Susskind) ('78)

(A) $D = 1+1$

$$H = -\sum_{j=1}^{N} \sigma_1(j) - \lambda \sum_{j=1}^{N} \sigma_3(j)\,\sigma_3(j+1)$$

(with PBC's and N even)



$\tilde{j} \equiv$ "$j + \frac{1}{2}$"

$$\tau_3(\tilde{j}) = \prod_{i \leq j} \sigma_1(i)$$ (operator that creates a kink)

$$\tau_1(\tilde{j}) = \sigma_3(j)\,\sigma_3(j+1)$$

check $\quad \tau_3(\tilde{j})^2 = I = \tau_1(\tilde{j})^2$

$$\{\tau_1(\tilde{j}), \tau_3(\tilde{j})\} = 0 \quad \text{anti-commute}$$

$$[\tau_1(\tilde{j}), \tau_3(\tilde{j}')] = 0 \quad \text{commute}$$

Dual Pauli Algebra

$$\Rightarrow H = -\sum_{\tilde{j}=1}^{N} \tau_3(\tilde{j})\,\tau_3(\tilde{j}+1) - \lambda \sum_{\tilde{j}=1}^{N} \tau_1(\tilde{j})$$

It is the same with $\lambda \leftrightarrow \frac{1}{\lambda}$!

self-dual at $\lambda_c = 1$

PBC's $\Rightarrow \sigma_3(N)\,\sigma_3(1) = \prod\limits_{\underset{\text{all}}{\tilde{j}}} \tau_1(\tilde{j}) = Q$

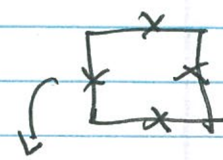
$\Rightarrow \sigma_3(N) = \sigma_3(1)$ iff $Q = 1$ (<u>invariant state</u>)

<u>Note</u>

In the ordered phase $\langle \tau_3(\tilde{j}) \rangle = 0$

In the disordered phase $\langle \tau_3(\tilde{j}) \rangle \neq 0$

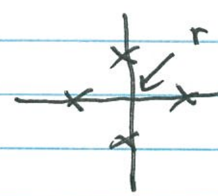
$\Rightarrow$ <u>Disorder Operator</u> (Kadanoff-Ceva '71, FS '78)



B. <u>$D = 2+1$</u>

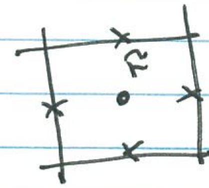
$$H = -\sum_{(r,\mu)} \sigma^1_\mu(r) - \lambda \sum_{(r,\mu\nu)} \sigma^3_\mu(r)\,\sigma^3_\nu(r)\,\sigma^3_\mu(r)\,\sigma^3_\nu(r)$$

$$Q(r) = \prod_{\substack{\text{at} \\ \text{a site}}} \sigma^1_\mu(r)$$

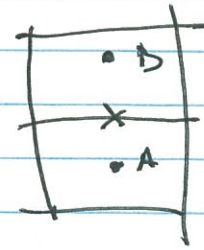


<u>Duality</u>

$$\underset{\substack{\uparrow \\ \text{on a} \\ \text{plaquette}}}{\sigma^3_\mu\,\sigma^3_\nu\,\sigma^3_\mu\,\sigma^3_\nu} \equiv \underset{\substack{\uparrow \\ \text{dual} \\ \text{site}}}{\tau_1(\tilde{r})}$$



$$\underset{\substack{\uparrow \\ \text{on} \\ \text{a link}}}{\sigma^1_\mu(r)} = \tau_3(A)\,\tau_3(B)$$

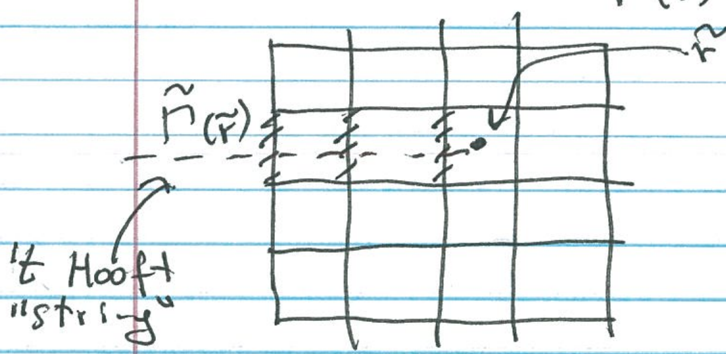


Note $\prod_{\text{vertex}} \sigma^1_\mu = \prod_{\square} \tau_3^2 = 1$

$\square$ ← dual

only gauge-invariant states!

Def $\tau_3(\tilde r) = \prod_{\tilde\Gamma(\tilde r)} \sigma^1_\mu(r) \Rightarrow \tau_3(\tilde r)\,\tau_3(\tilde r+1) = \sigma^1_\mu(r)$



Again $\tau_3^2 = \tau_1^2 = I$

$\{\tau_3, \tau_1\} = 0$ same dual site

or $[\tau_3, \tau_1] = 0$ (otherwise)

* In the confined phase

$$\langle \tau_3(\tilde r)\rangle = \left\langle \prod_{\tilde\Gamma(\tilde r)} \sigma^1_\mu(r) \right\rangle \neq 0$$

* In the deconfined phase

$\tau_3(\tilde r)$ creates a $\mathbb{Z}_2$ magnetic flux at $\tilde r$ ("vison")

In the Toric Code this state is called $\underline{\underline{m}}$

The state created by $Q = -1$ (e.g. Wilson arc) is called $\underset{\smile}{e}$

duality $e \leftrightarrow m$

$\Rightarrow$ In $D = 2+1$

$$H_{gauge} = - \sum_{link} \sigma^1_\mu - \lambda \sum_{plaquette} \prod \sigma_3$$

$\left(Q(r) = I \quad \text{everywhere} \right)$

Dual

$$H_{Ising} = - \sum_{\langle \tilde{r}, \tilde{r}' \rangle} \tau_3(\tilde{r})\, \tau_3(\tilde{r}') - \lambda \sum_{\tilde{r}} \tau_1(\tilde{r})$$

$$\lambda \leftrightarrow \frac{1}{\lambda}$$

* In $D = 3+1$ The gauge theory is self-dual

It has a 1st order transition at

$\lambda_c = 1$ ("Onsager")

The $\lambda \gg 1$ (deconfined) phase is topological (Wilson loops + 't Hooft surfaces)

$\lambda \ll 1$ confinement.