L 7 9/13/2022

Majorana Fermions (Jordan-Wigner transf.)

(I) $D = 1+1$

$$H = -\sum_j \sigma_1(j) + \lambda \sum_j \sigma_3(j)\sigma_3(j+1)$$

Def: $\chi_1(j) \equiv \tau_3(\tilde{j}-1)\,\sigma_3(j)$ Majorana fermions

$$= \Big(\prod_{k<j} \sigma_1(k)\Big)\sigma_3(j)$$

$$\chi_2(j) \equiv i\,\tau_3(\tilde{j})\,\sigma_3(j)$$

$$= \Big(i\prod_{k\le j} \sigma_1(k)\Big)\sigma_3(j)$$

Hermitian

$$\chi_1^2(j) = \chi_2^2(j) = 1 \;;\; \chi_1^\dagger(j) = \chi_1(j) \;;\; \chi_2^\dagger(j) = \chi_2(j)$$

and $\{\chi_a(j), \chi_b(j')\} = \delta_{ab}\,\delta_{jj'}$

Note: we can define $\psi(j) = \frac{1}{2}\big(\chi_1(j) + i\chi_2(j)\big)$

$$\psi^\dagger(j) = \frac{1}{2}\big(\chi_1(j) - i\chi_2(j)\big)$$

and $\{\psi(j), \psi(j')\} = 0$

$\{\psi(j), \psi^\dagger(j')\} = \delta_{jj'}$ (Dirac fermions)

Fermions $\simeq$ domain walls

(N even)

$$\Rightarrow H = -N + \sum_{j=-\frac{N}{2}+1}^{\frac{N}{2}} 2\psi^{\dagger}(j)\psi(j) +$$

$$- \lambda \sum_{j=-\frac{N}{2}+1}^{\frac{N}{2}} \left(\psi^{\dagger}(j) - \psi(j)\right)\left(\psi^{\dagger}(j+1) + \psi^{\dagger}(j)\right)$$

$+$ boundary term

boundary ~~term~~ term: $-\lambda\eta\,\sigma_3(\frac{N}{2})\,\sigma_3(-\frac{N}{2}+1) =$

$$= -\lambda\eta\,\hat{Q}\left(\psi^{\dagger}(\tfrac{N}{2}) - \psi(\tfrac{N}{2})\right)\left(\psi^{\dagger}(-\tfrac{N}{2}+1) + \psi(-\tfrac{N}{2}+1)\right)$$

$$\hat{Q} = \prod_{j=-\frac{N}{2}+1}^{N/2} \sigma_1(j) \quad \text{(spin flip operator)}$$

$$\equiv e^{i\pi\sum_j \psi^{\dagger}(j)\psi(j)} \equiv e^{i\pi \hat{N}_F} \equiv (-1)^{N_F}$$

$\hat{N}_F$: fermion #

$\Rightarrow$ $\hat{Q}$ is the fermion parity operator

$\eta = \pm$ for periodic / antiperiodic BC's

$\Rightarrow$ if the spins obey PBC's (APBC's)

$$\sigma_3(\tfrac{N}{2}+1) = \eta\,\sigma_3(-\tfrac{N}{2}+1) \Rightarrow \psi(\tfrac{N}{2}+1) = Q\,\eta\,\psi(-\tfrac{N}{2}+1)$$

If the spins have $\eta = +1$ (PBC's)

$\Rightarrow$ the fermion BC's depend on the fermion parity

In general the G.S. is in the fermion parity even sector

Fourier Transform (parity even sector)

$$\psi(j) = \frac{1}{N} \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} e^{i 2\pi \frac{kj}{N}} a(k)$$

$$a(k) = \sum_{j=-\frac{N}{2}+1}^{N/2} e^{-i 2\pi \frac{kj}{N}} \psi(j)$$

$$\{a(k), a^{\dagger}(k')\} = N \delta_{k,k'}$$

$$\{a(k), a(k')\} = 0$$

Thermodynamic limit: $N \to \infty$

$$\frac{2\pi}{N} k \equiv k \in (-\pi, \pi] \quad \text{(first BZ)}$$

$$\delta(k) = \lim_{N\to\infty} \frac{1}{2\pi} \sum_{j=-\frac{N}{2}+1}^{\frac{N}{2}} e^{ikj} = \lim_{N\to\infty} \frac{N}{2\pi} \delta_{k,0}$$

Dirac

periodic

$$\Rightarrow \{a(k), a^{\dagger}(k')\} = 2\pi\, \delta(k-k')$$

$$H = -N + 2\int_{-\pi}^{\pi} \frac{dk}{2\pi} (1-\lambda\cos k)\, a^{\dagger}(k) a(k)$$
$$- \lambda \int_{-\pi}^{\pi} \frac{dk}{2\pi} \left(e^{ik} a^{\dagger}(k) a^{\dagger}(-k) - e^{-ik} a(k) a(-k) \right)$$

Same Hamiltonian as that of the BCS theory of a p-wave superconductor in 1D!

The spectrum is obtained by a Bogoliubov Transf.

$$\omega(k) = 2\sqrt{1+\lambda^2 - 2\lambda\cos k} \geq 0$$

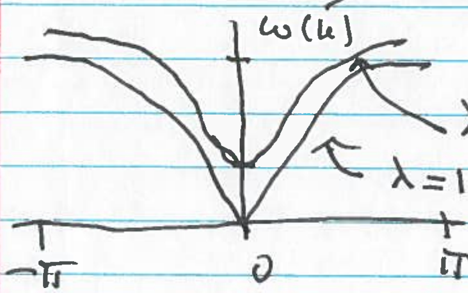
$$E_{gnd}(\lambda) \equiv \mathcal{E}_0(\lambda)\, N$$

$$\mathcal{E}_0(\lambda) = -\frac{1}{2}\int_{-\pi}^{\pi} \frac{dk}{2\pi}\, \omega(k) < 0$$

$$\omega(k) = 2\left((1-\lambda)^2 + 4\lambda \sin^2(k/2) \right)^{1/2}$$

<u>Phase Transition</u>

At $\lambda = 1 \Rightarrow \omega(k) = 4\,|\sin k/2|$



$\text{gap} = \omega(0) = 2\,|1-\lambda|$

$\Rightarrow \nu = 1$ and $z = 1$

$\mathcal{E}_0(\lambda)$ is continuous but has a singularity

$$\frac{\partial^2 \mathcal{E}_0}{\partial\lambda^2} \cong -\frac{1}{2\pi} \ln\left(\frac{8}{|1-\lambda|}\right) + \ldots$$

same as the singularity of the heat capacity at T_c of the 2D Model.

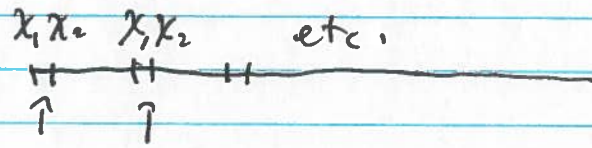
Back to the Majorana picture

$$H = i\sum_j \chi_1(j)\,\chi_2(j) + i\lambda \sum_j \chi_2(j)\,\chi_1(j+1)$$

Consider an <u>open</u> <u>chain</u> with even Fermion parity

Note that at <u>every site</u> I have a χ_1 and a χ_2

It will be convenient to split them



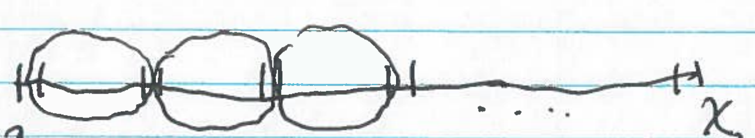
Two limits

① $\lambda \to 0$ (<u>disordered</u> phase of the spins)

$$\Rightarrow H \simeq i\sum_j \chi_1(j)\,\chi_2(j)$$

$\Rightarrow$ G.S. is unique

② $\lambda \to \infty$ (<u>ordered</u> phase of the spins)



χ_1 does not enter! does not enter!

In this phase we have two operators

$\chi_1(-\frac{N}{2}+1)$ and $\chi_2(\frac{N}{2})$ that

commute with H!

Majorana zero modes

i.e $1/2$ of the fermion is at $-\frac{N}{2}+1$

and the other $1/2$ at $\frac{N}{2}$!

One can prove that this is true

for $\lambda > 1$ (not just $\lambda \to \infty$)

These zero modes cannot be removed!

Topological protection.

Basis of "Majorana" qubits.

$\Rightarrow$ In the fermion basis the ordered phase

of the 1D chain is topological but not

in the spin basis

Reason: the JW mapping is not one to one

but two to one

This is only apparent by changing BC's

(CF, M. Srednicki, L. Susskind, PRD 1980)

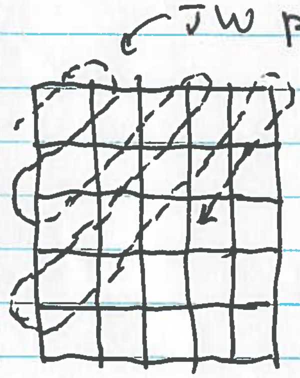
Does this work in dimensions $d > 1$?

The answer is <u>not quite</u>

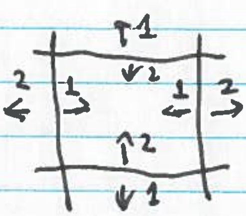
Consider the $\mathbb{Z}_2$ gauge theory in 2+1 Dimensions

$$H = -\sum_{(r,\mu)} \sigma^1_\mu(r) - \lambda \sum_{(r,\mu\nu)} \sigma^3_\mu(r)\, \sigma^3_\nu(r+e_\mu)\, \sigma^3_\mu(r+e_\nu)\, \sigma^3_\nu(r)$$

We will use a Jordan-Wigner transf



Recall that in the 1+1 Ising Model we had <u>two</u> Majoranas for each site.
Now we have two Majoranas for <u>each link</u>



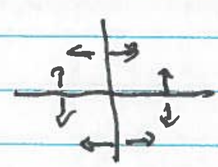
$$H = -i \sum_{(r,\mu)} \psi^{(1)}_\mu(r)\, \psi^{(2)}_\mu(r)$$

$$+ \lambda \sum_{(r,\mu\nu)} \psi^{(2)}_\mu(r)\, \psi^{(1)}_\nu(r+e_\mu)\, \psi^{(2)}_\mu(r+e_\nu)\, \psi^{(1)}_\nu(r)$$

("fermionic string")

(this can also ~~be~~ done using Grassmann variables)

Note: gauge-invariance $\Rightarrow Q(r) = \prod_{\text{vertex}} \sigma_1 = 1$

$\Rightarrow$  $= 1$

The string defines ~~the~~ fermion operators in terms of the σ's $\Rightarrow$ non-local

However $\{\varphi(r), \psi_\mu^{1,2}(r)\} = 0$

$\Rightarrow$ we cannot have an isolated fermion (not gauge-invariant)

In the confining phase, $\lambda < \lambda_c$, the fermions are confined $\Rightarrow$ the energy $\sim$ distance

In the deconfined phase $\lambda > \lambda_c$ the fermions are deconfined

$\Rightarrow$ the fermion is a finite energy excitation of the topological phase

$\Rightarrow$ In the topological phase the

(1) gnd state is degenerate on a closed surface

(2) The spectrum has 4 particles

(1) I (nothing)

(2) e (charge)

(3) m (flux)

(4) ψ fermion

Similar construction in 3+1 dimensions

(Srednicki, PRD 1980)