Frustration, Quantum Disorder and Gauge Theory

(ER: FT CMP, Ch 8)

Back to the QH AFM \( H = \sum_{\langle r, r' \rangle} \mathbf{S}(r) \cdot \mathbf{S}(r') \)

Square lattice

\[ \mathbf{\vec{Q}} = (\pi, \pi) \]

\[ \langle \hat{N}(\mathbf{r}) \rangle = (-1)^{x+y} \langle \mathbf{S}(\mathbf{r}) \rangle \quad \text{(non-linear O-model)} \]

\[ \begin{array}{c}
J_1 > 0, \quad J_2 > 0 \\
\text{If } J_1 \approx J_2 \text{ frustration}
\end{array} \]

Classically degenerate state,

\[ \begin{array}{c}
J > 0 \\
\text{If the Néel state can be destroyed}
\end{array} \]

by quantum fluctuations \( \Rightarrow \) Spin singlet state

Simple model: "Valence bonds" (Anderson ~ 1973)

\[ |(\xi, j)\rangle = \frac{1}{\sqrt{2}} (|\uparrow i, \downarrow j\rangle - |\downarrow i, \uparrow j\rangle) \]

\[ |\mathbb{VB}\rangle = \prod_{\text{any}} |(\xi, j)\rangle \]
Resonating Valence Bond state ("RVB")

\[ |\Psi\rangle = \sum \prod_{\text{pairs}} \alpha (\bar{\psi}_k \delta_k) |\psi_k \rangle \text{ permutations} \]

\[ \alpha (\bar{\psi}_k \delta_k) = \alpha (1 \bar{\psi}_k - \delta_k) \]

Singlets with same distance are in superposition ("resonate")

However: If \( \alpha (|1x1\rangle) \sim \frac{\#}{\text{large } |1x1\rangle} \) \( \delta < 5 \)

\[ \text{This is the same as Néel!} \]

Short-range RVB state

\( |\text{SR RVB} \rangle = \sum (|111\rangle + |1\cdots\rangle) \)

Plaquettes

This state has no long-range order of spins

If we make the approximation that there is no overlap between \( |111\rangle \) and \( |\cdots\rangle \) (some with other configs that share a site)
Valence bonds behave as classical dimers

\[ \langle VB | VB \rangle = \sum_{SR} \frac{1}{c} \text{ classical dimer config} \]

We will see that states of this type are critical on bipartite lattices (e.g., square) and topological on non-bipartite lattices (e.g., triangular).

Spinons, holons, and VB states can represent angular momentum operators (spins) in \( \pm 1 \) ways. For example, if \( \hat{S} \) are the three spin \( \frac{1}{2} \) operators we can represent them in terms of a set of fermions

\[ \hat{S}(x) = \frac{1}{2} \sum_{x} \bar{C}_{\alpha}(x) \sigma_{\alpha \beta} C_{\beta}(x) \]

Pauli (generators of \( SU(2) \)) in the \( S=\frac{1}{2} \) rep.
but for $S=1/2$ we have only two spin states, $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$

and two fermions have 4 states $|10\rangle, |1\uparrow\rangle, |1\downarrow\rangle, |1\uparrow\downarrow\rangle$

We need to impose a constraint to restrict the Hilbert space

$n(x) \equiv \sum_{\alpha \beta} a_\alpha^\dagger(x) a_\beta(x) = 1 \quad \text{(Single occupancy)}$

summed over

Alternative: "Schwinger bosons"

$S(x) = \frac{1}{2} \sum_{\alpha \beta} \bar{a}_\alpha(x) \bar{a}_\beta(x)$

$\Rightarrow a_\alpha^\dagger(x) a_\beta(x) = 1 \quad \text{("hard core")}$

$|\psi(c, i\hat{x})\rangle \equiv \sum_{\alpha \beta} c_\alpha(x) a_\beta^\dagger(c) |10\rangle$

$\uparrow$ empty

$\equiv (c^\dagger_1(x_1) c^\dagger_i(y_i) - c^\dagger_i(y_i) c^\dagger_1(x_1)) |10\rangle$

which is "maximally entangled state"
Charge: add "holes"

\[ b(x); \text{ boson} \]

\[ f^a(x); \text{ fermion} \]

s.t. \[ b^+(x) b(x) + f^+_a(x) f^a(x) = 1 \]

\[ \implies \text{ each site is either empty (a "hole") or occupied by a spin} \]

\[ \implies \text{ no doubly occupied sites, (strong local repulsion)} \]

holon \[ |n\rangle \equiv |e, 0\rangle = b^+|0\rangle \]

"spinons" \[ \begin{align*}
|1\rangle & \equiv |0, 1\rangle = f^+_0|0\rangle \\
|1\rangle & \equiv |0, 0\rangle = f^+_1|0\rangle 
\end{align*} \]

Formally

\[ c^+_0(x) \equiv b(x) f^+_0(x) \]
Gauge theory picture

\[ S(x), \bar{S}(y) = \frac{1}{2} \sum_{\alpha, \alpha'} C_{\alpha}(x) \bar{C}_{\alpha'}(x) C_{\alpha}(y) \bar{C}_{\alpha'}(y) \]

\[-\frac{1}{4} \eta(x) \eta(y) \]

(\text{I recall that } \frac{\delta}{\delta \eta} \delta \bar{S} = 2 \eta x \delta x - \delta \eta \delta \bar{S})

\text{Constraint: } \eta(x) = 1

\Rightarrow H = \sum_{x} \frac{1}{2} \left( \sum_{\alpha} C_{\alpha}^{\dagger}(x) \bar{C}_{\alpha}(x) C_{\alpha}(x + r_{y}) \right) \cdot \bar{C}_{\alpha}(x + r_{y})

\text{direction} + \text{constraint}

(square lattice)

Path integral over Grassmann variables

Lagrangian \quad \text{chemical potential}

\[ L = \sum_{x} \left( \sum_{\alpha} C_{\alpha}^{\dagger}(x, t) \left[ i \gamma \cdot \partial - \mu \right] C_{\alpha}(x, t) \right) \]

\[ + \sum_{x} \phi(x, t) \left( C_{\alpha}(x, t) \bar{C}_{\alpha}(x, t) - 1 \right) - H \]

Lagrange multi. field

\text{that enforces the constraint}
Since $H$ is quartic on fermions $\Rightarrow$ Hubbard

- Stratonovich decoupling

The HS fields live on links ("VB's")

$$X^*_{x,y}(x,t) = X^*_{x,y}(x+\epsilon y, t)$$

Complex fields on the links

$$L^1 = \sum_{x} \sum_{\alpha} \bar{c}_{\alpha}^x(x,t)(i\Sigma + \Phi(x,t))c_{\alpha}^x(x,t) + \sum_{x} \Phi(x,t)(c_{\alpha}^x(x,t)c_{\alpha}^x(x+t \epsilon y, t) - 1)$$

$$- \frac{1}{2} \sum_{(x,y)} \left| X^*_{x,y}(x,t) \right|^2$$

$$+ \sum_{(x,y)} (\bar{c}_{\alpha}^x(x,t)X^*_{x,y}(x,y) c_{\alpha}^y(x+y \epsilon y, t)$$

$$+ c_{\alpha}^x(x+y \epsilon y, t)X^*_{x,y}(x,y) c_{\alpha}^x(x,t)$$

This theory has a $U(1)$ gauge invariance!

$$c_{\alpha}^x(x,t) = e^{i\phi^x(x,t)}c_{\alpha}^x(x,t)$$

$$\Phi(x,t) = \phi'(x,t) + \frac{e}{\epsilon} \wedge (x,t)$$

$$X^*_{x,y}(x,t) = e^{-i\phi \wedge (x,t)}X^*_{x,y}(x+t \epsilon y, t)$$

$\Rightarrow \phi$ is the true component of a $2+1$ dim. $U(1)$ gauge field.
Write \( X_j^i(x,t) = \xi_j^i(x,t) e^{-i\mathbf{A}^j(x,t)} \)

where \( \xi_j^i(x,t) \neq 0 \)

\[
\Rightarrow \mathbf{A}^j(x,t) = \mathbf{A}^j(x,t) + \mathbf{\Lambda}(x+\text{eq}_{j},t) - \mathbf{\Lambda}(x,t)
\]

\( \mathbf{A}^j \cdot \mathbf{\Lambda}(x,t) \)

\[\text{This looks like electromagnetism} \]

but \( \mathbf{A}^j \rightarrow \mathbf{A}^j + 2\pi \mathbf{\eta}_j^i(x) \) is an exact symmetry & periodicity!

\[
Z = \int \mathcal{D}c^{2k} \mathcal{D}x^i \mathcal{D}x_j \ e^{i\mathbf{S}} \\
\text{is a trace} \]

\[
S = \int dt \mathcal{L} \\
\text{under a gauge transform:} \]

\[
S \rightarrow S - \sum_x \int dt \ \delta_x \mathbf{\Lambda}(x,t) \]

\[= S - \sum_x \left( \mathbf{\Lambda}(x,t+\infty) - \mathbf{\Lambda}(x,t) \right) \]

Since \( Z \) is a trace \( \Rightarrow \) PBC's in time.
To proceed further we need to make approximations. But, this theory does not have a small parameter.

Use a large-$N$ approach (Affleck, Marston 1988; Reall, Sachdev 1988)

Let $x = 1, \ldots, N$ ($N=2$ for SU(2))

We now have an $SU(N)$ symmetry

$\text{Constraint} \sum_{a=1}^{N} c_{a}^{+}(x) c_{a}(x) = \delta \cdot n(x)$

For a bipartite lattice we have two options

**A** $n(x) = \begin{cases} 1 & \text{if } x \text{ is in sublattice } A \\ N-1 & \text{if } x \text{ is in sublattice } B \end{cases}$

Problem: This choice breaks symmetries (e.g., translation and/or point group symmetries)

**B** $n(x) = \frac{N}{2}$ irrespective of whether the sublattice (self-conjugate rep)

Restriction: $N$ must be even

Fundamental rep: choose $\frac{N-1}{2}$ symplectic

Another option: use the group $Sp(N)$: group of
$2N \times 2N$ unitary matrices that leave invariant the valence 2nd operator 
\[ J_{\alpha\alpha'} \rightarrow C_{\alpha} \sigma C_{\alpha'} \sigma' \]
where \( J_{\alpha\alpha'} = \delta_{\alpha\alpha'} \delta_{\sigma\sigma'} \), \( \sigma, \sigma' = 1, 2 \) and \( \alpha, \alpha' = 1, 2, \ldots, N \).

\( \text{Sp}(1) \cong SU(2) \) + a constraint that fixes the representation at site \( x \).

We will use the Affleck-Narison approach with self-conjugate reps., i.e., \( N(x) = \frac{N}{2} \).

The Lagrangian now is:
\[ L = \sum_{\alpha \beta} C_{x}^{\dagger}(x, t) \left[ \left( \partial_{t} + \mu \right) C_{x}^{\dagger}(x, t) + \phi \right] C_{x}^{\dagger}(x, t) C_{x}(x, t) - \frac{1}{2} \]
\[ - \frac{N}{2} \left| \sum_{x} X_{ab}^{\dagger}(x, t) \right|^{2} + \]
\[ + \frac{1}{2} C_{\alpha}^{\dagger}(x, t) X_{ab}^{\dagger}(x, t) C_{\alpha}(x + e_{i}, t) + \phi, \]
\[ + C_{\alpha b}^{\dagger}(x + e_{i}, t) X_{ab}(x, t) C_{\alpha b}^{\dagger}(x, t) \]
\[ \sum_{x} X_{ab}^{\dagger}(x, t) \text{ is an } n_{c} \times n_{c} \text{ complex matrix } \]
\[ X_{ab}^{\dagger}(x, t) = X_{ab}^{\dagger}(x, t) \]
\[ a, b = 1, \ldots, n_{c}, \quad \alpha, \beta = 1, \ldots, N \]