L 8 9/15/2022

Frustration, Quantum Disorder and Gauge Theory

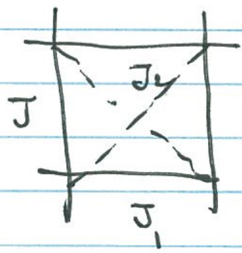
(EF: FTCMP, Ch 8)

Back ~~to~~ to the QHAFM $H = J \sum_{\langle r, r' \rangle} \vec{S}(r) \cdot \vec{S}(\vec{r}')$

square lattice

Néel state
$\vec{Q} = (\pi, \pi)$

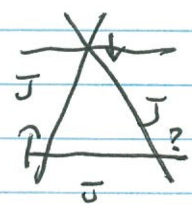
$\langle \vec{N}(\vec{r}) \rangle = (-1)^{x+y} \langle \vec{S}(\vec{r}) \rangle$ (non-linear σ-model etc.)



$J_1 > 0, \; J_2 > 0$

If $J_1 \approx J_2$ frustration

Classically degenerate states



$J > 0$

If the Néel state ~~can~~ could be destroyed by quantum fluctuations $\Rightarrow$ ~~spin~~ Spin singlet state

Simple model; "valence bonds" (Anderson ~ 1973)

$$|(i,j)\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow_i \downarrow_j\rangle - |\downarrow_i \uparrow_j\rangle \right)$$

$$|VB\rangle = \prod_{\text{pairs}} |(i_k, j_k)\rangle \quad ?$$

Resonating Valence Bond state ("RVB")

$$|\Psi\rangle = \sum_{\text{permutations}} \prod_{\text{pairs}} a(i_k, j_k)\, |(i_k, j_k)\rangle$$

$$a(i_k, j_k) = a(|i_k - j_k|)$$

singlets with same distance are in superposition ("resonate")

However: If $a(|x|) \sim \dfrac{\#}{|x|^\sigma}$ (large $|x|$) $\sigma < 5$

This is the <u>same</u> as Néel !

<u>Short range RVB state</u>

~~SRRVB~~

VB's

$$|RVB\rangle_{SR} = \sum_{\text{plaquettes}} \left(|\,||\,\rangle + |=\rangle \right)$$

This state has no long-range order of spins

If we ~~assu~~ make the approximation that there is <u>no overlap</u> between $|\,||\,\rangle$ and $|=\rangle$ (same with other configs that share a site) $\Rightarrow$

$\Rightarrow$ Valence bonds behave as classical dimers

and

$$\langle VB | VB \rangle_{SR} = \sum_{\substack{\text{classical} \\ \text{dimer} \\ \text{configs}}} 1$$

We will see that state of this type are critical on bipartite lattices (e.g. square) and topological on non-bipartite lattices (e.g. triangular)

Spinons, holons and VB states

we can represent angular momentum operators (spins) in $\neq$ ways.

For example, if $\vec{S}$ are the three spin 1/2 operators we can represent them in terms of a set of fermions

$$\vec{S}(\vec{x}) = \frac{1}{2} c^{\dagger}_{\alpha}(x) \, \vec{\sigma}_{\alpha\beta} \, c_{\beta}(x)$$

Pauli (generators of SU(2) in the S = 1/2 rep)

<u>but</u> for $S = 1/2$ we have only <u>two</u> spin states, $|\uparrow\rangle$ and $|\downarrow\rangle$

and two fermions have <u>4 states</u>

$|0\rangle, \ |\uparrow\rangle, \ |\downarrow\rangle, \ |\uparrow\downarrow\rangle$

We need to impose a <u>constraint</u> to restrict the Hilbert space

$$n(x) \equiv \underset{\substack{\uparrow \\ \text{summed over}}}{c^\dagger_\alpha(x)\, c_\alpha(x)} = 1 \qquad \text{(single occupancy)}$$

<u>Alternative</u>: "Schwinger bosons"

$$\vec{S}(x) = \frac{1}{2}\, a^\dagger_\alpha(x)\, \vec{\sigma}_{\alpha\beta}\, a_\beta(x)$$

$$\Rightarrow \quad a^\dagger(x)\, a(x) = 1 \qquad (\text{"hard core"})$$

$$|(i,j)\rangle \equiv \varepsilon_{\alpha\beta}\, c^\dagger_\alpha(i)\, c^\dagger_\beta(j)\, \underset{\substack{\uparrow \\ \text{empty}}}{|0\rangle}$$

$$\equiv \left(c^\dagger_\uparrow(i)\, c^\dagger_\downarrow(j) - c^\dagger_\downarrow(i)\, c^\dagger_\uparrow(j) \right) |0\rangle$$

"maximally entangled state"

Charge : add "holes"

$b(x)$: boson

$f_\alpha(x)$: fermion

s.t. $b^\dagger(x)\, b(x) + f_\alpha^\dagger(x)\, f_\alpha(x) = 1$

$\Rightarrow$ each site is either empty (a "hole")

or ~~occupp~~ occupied by a spin

$\Rightarrow$ no doubly occupied sites.

(strong local repulsion)

holon $|h\rangle \equiv |e,0\rangle \equiv b^\dagger |0\rangle$ spinless boson

"spinons" $\left\{ \begin{array}{l} |\uparrow\rangle \equiv |0,\uparrow\rangle \equiv f_\uparrow^\dagger |0\rangle \\ |\downarrow\rangle \equiv |0,\downarrow\rangle \equiv f_\downarrow^\dagger |0\rangle \end{array} \right\}$ $s=1/2$ fermions

Formally

$$c_\sigma^\dagger(x) \equiv b(x)\, f_\sigma^\dagger(x)$$

Gauge theory picture

$$\vec{S}(x) \cdot \vec{S}(y) = \frac{1}{2} c^\dagger_\alpha(x) c_\beta(x) c^\dagger_\beta(y) c_\alpha(y)$$
$$- \frac{1}{4} n(x) n(y)$$

(I used that $\vec{\sigma}_{\alpha\beta} \cdot \vec{\sigma}_{\gamma\delta} = 2 \delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\gamma\delta}$)

Constraint: $n(x) = 1$

$$\Rightarrow H = \frac{J}{2} \sum_{(x, j)} c^\dagger_\alpha(x) c_\beta(x) c^\dagger_\beta(x+e_j) c_\alpha(x+e_j)$$

direction + constraint

(square lattice)

Path-integral over Grassmann variables

Lagrangian chemical potential

$$L = \sum_x c^\dagger_\alpha(x,t) (i\partial_t + \mu) c_\alpha(x,t)$$
$$+ \sum_x \phi(x,t)\left(c^\dagger_\alpha(x,t) c_\alpha(x,t) - 1\right) - H$$

Lagrange mult. field that enforces the constraint

Since H is quartic in fermions $\Rightarrow$ Hubbard - Stratonovich decoupling

The HS fields live on links ("VB's")

$$\chi_j(x,t) = \chi^*_{-j}(x+e_j,t)$$

complex fields on the links

$$L' = \sum_x \underset{\substack{\uparrow \\ \text{Grassmann} \\ \text{variables}}}{c^\dagger_\alpha(x,t)} (i\partial_t + \mu) c_\alpha(x) + \sum_x \phi(x,t)\left(c^\dagger_\alpha(x,t)c_\alpha(x,t) - 1\right)$$

$$- \frac{2}{J} \sum_{\substack{(x,j) \\ \uparrow \text{links}}} |\chi_j(x,t)|^2$$

$$+ \sum_{(x,j)} \left(c^\dagger_\alpha(x,t)\, \chi_j(x,t)\, c_\alpha(x+e_j,t) + c^\dagger_\alpha(x+e_j,t)\, \chi^*_j(x,t)\, c_\alpha(x,t)\right)$$

This theory has a U(1) gauge invariance!

$$c_\alpha(x,t) = e^{i\Lambda(x,t)}\, c'_\alpha(x,t)$$

$$\phi(x,t) = \phi'(x,t) + \partial_t \Lambda(x,t)$$

$$\chi_j(x,t) = e^{-i\Lambda(x,t)}\, \chi'_j(x,t)\, e^{i\Lambda(x+e_j,t)}$$

$\Rightarrow$ ϕ is the time component of a 2+1 dim. U(1) gauge field

Note:

Write $\chi_j(x,t) \equiv \rho_j(x,t)\, e^{iA_j(x,t)}$

where $\rho_j(x,t) \geqslant 0$

$\Rightarrow A_j(x,t) = A_j'(x,t) + \underbrace{\Lambda(x+e_j,t) - \Lambda(x,t)}_{\Delta_j \Lambda(x,t)}$

This looks like electromagnetism

but $A_j \to A_j + 2\pi n_j(x)$ is

an exact symmetry $\Leftrightarrow$ periodicity!

$$Z = \int \mathcal{D}c^\dagger \mathcal{D}c\, \mathcal{D}\chi_j^\dagger \mathcal{D}\chi_j\, e^{i\underset{\text{action}}{S}}$$

is a trace

$$S = \int dt\, L$$

under a gauge transf.

$$S \to S - \sum_x \int dt\, \partial_t \Lambda(x,t)$$

$$= S - \sum_x \left(\Lambda(x, t\to+\infty) - \Lambda(x, t\to-\infty)\right)$$

Since Z is a trace $\Rightarrow$ PBC's in time.

To proceed further we need to make approximations. But, this theory does not have a small parameter.

use a large-N approach (Affleck, Marston 1988
Read, Sachdev 1988)

Let $\alpha = 1, \ldots, N$ ($N=2$ for $SU(2)$)

we now have an $SU(N)$ symmetry

Constraint $\sum_{\alpha=1}^{N} c^{\dagger}_{\alpha}(x)\, c_{\alpha}(x) = n(x)$

For a bipartite lattice we have two options

(A) $n(x) = \begin{cases} 1 & \text{if } x \text{ is in sublattice A} \\ N-1 & \text{if } x \text{ is in sublattice B} \end{cases}$

Problem: this choice breaks symmetries (e.g. translation and/or point group symmetries)

(B) $n(x) = \frac{N}{2}$ irrespective of ~~whether~~ the sublattice (self-conjugate rep)

Restriction: N must be even

Fundamental Rep: choose $n_c = 1$

Another option: use the group $Sp(N)$: group of symplectic

$2N \times 2N$ unitary matrices that leave invariant the valence bond operator $J^{aa'}_{\sigma\sigma'} \, c^{\dagger}_{ia\sigma} c^{\dagger}_{ja'\sigma'}$

where $J^{aa'}_{\sigma\sigma'} = \delta_{aa'} \, \varepsilon_{\sigma\sigma'}$, $\sigma, \sigma' = 1, 2$ and $a, a' = 1, \ldots, N$

$\left(Sp(1) \cong SU(2) \right)$ + a constraint that fixes the representation at site x.

We will use the Affleck–Marston approach with self-conjugate reps., i.e. $n(x) = \frac{N}{2}$

The Lagrangian now is

$$\mathcal{L} = c^{\dagger}_{\alpha a}(x,t)\left(i\partial_t + \mu\right) c_{\alpha a}(x,t) + \phi_{ab}(x,t)\left(c^{\dagger}_{\alpha a}(x,t)\, c_{\alpha b}(x,t) - \frac{N}{2}\delta_{ab}\right)$$

$$- \frac{N}{J} \left| \chi^{ab}_{j}(x,t) \right|^2 +$$

$$+ c^{\dagger}_{\alpha a}(x, t)\, \chi^{ab}_{j}(x,t)\, c_{\alpha b}(x+e_j, t) + \ldots$$

$$+ c^{\dagger}_{\alpha b}(x+e_j, t)\, \chi^{ab}_{j}(x,t)^{*}\, c_{\alpha a}(x,t)$$

$\chi^{ab}_{j}(x,t)$ is an $n_c \times n_c$ complex matrix

$$\chi^{ab}_{j}(x,t) = \chi^{ba}_{-j}(x+e_j, t)^{*}$$

$a, b = 1, \ldots, n_c$, $\alpha, \beta = 1, \ldots, N$