The Concept of Order in CMP

During the long history of Humanity many things have been called phases of matter (air, fire, water...).

Water presents itself in many "phases":

- liquid water, steam (vapor), ice (20 phases!)

![Phase diagram of water](image)

- ice are solid phases of water with a crystal structure
- liquid water and water vapor are not distinct phases
- "Line" of 1st order transitions (where the density jumps) ending at the critical point
Symmetry plays a key role in physics and in the phases of matter. In this class we will see that topology plays a role which is just as important.

Symmetry and phases of matter

Consider a simple model of a ferromagnet: The Ising model.

Square lattice with a spin at each site

$\sigma = +1 (\uparrow), -1 (\downarrow)$ (uniaxial)

$E(\sigma) = - J \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} \sigma(\mathbf{r}) \sigma(\mathbf{r}') - H \sum_{\mathbf{r}} \sigma(\mathbf{r})$

\[ Z = \sum_{[\sigma]} e^{-E(\sigma)/T} \]

$k_B = 1$

$N$ sites $\Rightarrow 2^N$ configurations.

*If $N$ is finite $\Rightarrow Z$ is an analytic function of $\frac{E}{T}$ and $\frac{H}{T}$.

*If $N \to \infty$ $\Rightarrow$ the physics depends on whether $H = 0$ or $H \neq 0$. 
\[
\langle \sigma \rangle = m \quad \text{(per site)}
\]

Order parameter

\[
\exp \chi = \frac{dM}{dT_H}
\]

This model also describes the liquid-gas transition and \( T_c \) is the critical exponent

\[ n: \text{density} \Rightarrow \Delta Q = \Delta n \quad \Delta n \Leftrightarrow m \]

What symmetry?

\[ \sigma(\hat{r}) \rightarrow -\sigma(\hat{r}) \quad \text{everywhere} \]

\( H \) breaks the symmetry (symmetry breaking)

\[
\text{Explicitly}
\]

However, if one cools below \( T_c \) and then \( H \rightarrow 0 \)

\[ \Rightarrow \langle m \rangle > 0 \quad \text{if} \ H \rightarrow 0^+ \quad \text{and} \ m < 0 \quad \text{if} \ H \rightarrow 0^- \]

Crucial: the thermodynamic limit first (\( N \rightarrow \infty \))
Why?
If \( N \to \infty \) and \( H > 0 \) \( \Rightarrow \) configs with \( n < 0 \) are exponentially suppressed (in \( W \))
(same with \( H < 0 \))
\( \Rightarrow T < T_c \Rightarrow \) the global \( Z_2 \) symmetry
is spontaneously broken
(\( \Rightarrow \) the state has less symmetry than
the Hamiltonian \( H \to 0 \)) \( \Rightarrow \) go to 4'
Other examples
\( x \) Spin system with easy plane anisotropy
\( x \) Superfluid and Superconductor
Order parameter is a complex \#
\[ \phi (\hat{r}) = |\phi (r)| e^{i \theta (r)} \]
\( \theta \) phase
amplitude
Also
\[ m_x = \text{Re } \phi \]
\[ m_y = \text{Im } \phi \]
Symmetry: \( \phi (\hat{r}) \to \phi (\hat{r}) e^{i \alpha} \)
\( \alpha \in [0, 2\pi) \) constant
(Global symmetry) \( U(1) \) symmetry
Typical con fig. domain walls.

For $T < T_c$ the domain walls are suppressed but at $T_c$ they proliferate.

Correlators

\[ \langle \sigma(x) \sigma(y) \rangle \sim \text{prob. to find two parallel spins at distance} \]

\[ R = |x-y| \]

$T > T_c$

\[ \langle \sigma(x) \sigma(y) \rangle \sim e^{-R/\xi} \]

$T < T_c$

\[ \langle \sigma(x) \sigma(y) \rangle \sim |\langle \sigma(x) \rangle|^2 + \# e^{-R/\xi} \text{(connected)} \]

\[ \xi \rightarrow 0 \text{ as } T \rightarrow T_c \]

\[ \xi \sim \frac{1}{|T-T_c|^\nu} \]
\[ |\Phi(z)| = \Phi_0 \text{ const.} \]

\[ \Rightarrow E[\Theta(z)] = -J_0 \sum \cos \left( \Theta(r) - \Theta(r') \right) \]

\[ (X\ Y \text{ model}) \ - \frac{1}{T} \sum_n \cos \Theta_n \]

not allowed in superfluids and superconductors!

\[ Z = \frac{1}{T} \int_0^{2\pi} d\Theta(r) \quad e^{-E[\Theta(r)]/T} \]

\[ \langle e^{i\Theta(z)} \rangle = 0 \quad \text{for } T > T_c \]

\[ \langle e^{i\Theta(z)} \rangle \equiv \Phi(z) \quad T < T_c \]

Note: the phase of \( \Phi(z) \) is arbitrary!

\[ \Rightarrow \text{SSB of } U(1) \text{ symmetry.} \]

Also \( T < T_c \), \( \Theta(r) \) varies slowly.

\[ \Rightarrow E[\Theta] \approx + J_0 \sum (\Theta(r) - \Theta(r'))^2 \]

\[ \text{"spin-wave approx"} \quad \frac{1}{2} \sum_{r,r'} \langle \Theta(r) \Theta(r') \rangle \quad \text{(periodicity?)} \]

\[ \approx \frac{J_0}{2\alpha^D} \int d^D \Phi \left( \alpha^D \Theta^2 \right) \]
Two symmetries:

- Global $U(1)$ symmetry \( \Theta(x) \rightarrow \Theta(x) + \alpha \)

- Local symmetry (Periodicity)
  \[ \Theta(x) \rightarrow \Theta(x) + 2\pi n(x) \]

\[
Z = \frac{1}{\mathcal{Z}} \prod_i \int \frac{d\Theta(x)}{2\pi} \sum_{\text{link}} e^{-\frac{\beta}{2} \sum_i (\Theta(x) - \Theta(x+\epsilon) + 2\pi n(x))}
\]

\[ e^{i\Theta(x)} \text{ is periodic but breaks } U(1) \]

\[ \Theta(x) \text{ breaks both } \]

Meaning of \( A(x) \)?

\[ \overline{e} \mu \rightarrow \Phi(x) \equiv p \]

\( \Theta(x) \) jumps by \( 2\pi p \) on these links
At long distances

Vortex phase winds by $2\pi p$

Draw a large circle $\Theta = \pi/2$

$r = 0$

$x - - - - - - - - - - - - - - - - C$

$\theta = 0$

$\frac{3\pi}{2}$

$\phi(r) \to 0 \text{ as } r \to 0$

$\phi(r) \to \phi_0 e^{i\Theta(r)}$ on $C$

On $C$ map of $C \to \Theta S_1$ circle defined by $\Theta(r)$ (mod $2\pi$)

$\Rightarrow p$ is the vorticity or winding $\#$
\( \left( \frac{\Delta \Theta}{2 \pi} \right)_C = \oint_{C} d\Phi = \oint_{C} \frac{e^{i \Theta(q)}}{2\pi} \, dq \)

\( \equiv p \)

\( p \) is a topological invariant

It does not change if \( C \) changes smoothly or if the config. changes smoothly.

Axial current  \( J = 1 \phi_0 \vec{\nabla} \Theta \)

Vorticity \( \omega(x) = \varepsilon_{\mu \nu} \partial_\mu \partial_\nu \Theta \)

If \( \Theta \) is smooth (holom. diff.)

\( \Rightarrow \varepsilon_{\mu \nu} \partial_\mu \partial_\nu \Theta = 0 \Rightarrow \omega = 0 \)

Vortex \( \omega \neq 0 \Rightarrow \text{cross derivatives do not commute at } x = 0 \)
Energy of a vortex line f's.

\[ \omega(x) = \sum_j \pi n_j \cdot \delta^2(x-x_j) \]

\[ \Rightarrow \Theta(x) = \sum_j \pi n_j \cdot \text{Im} \ln(z-z_j) \]

\[ z = x_1 + i x_2 \]

Cauchy-Riemann

\[ \frac{\partial \Theta}{\partial x} = \frac{\partial \Theta}{\partial y} \]

\[ \Rightarrow \Theta \text{ obeys } -\nabla^2 \Phi = \omega(x) \quad \text{(Poisson Eqn.)} \]

\[ E = \int \phi_0^2 \left( \frac{d^2\Theta}{dx} \right)^2 \]

\[ = \int \phi_0^2 \left( \frac{d^2\Theta}{dx} \right)^2 \]

\[ = -\int \phi_0^2 \left( \frac{d^2\Theta}{dx} \right)^2 \]

\[ = + \int \phi_0^2 \left( \frac{d^2\Theta}{dx} \right)^2 \]

\[ \text{over } \omega(x) \Theta(x) \]
Solve Poisson

$$\nabla^2 \varphi(x) = \int d^2 y \ G(x-y) \, \varphi(y)$$

$$- \nabla^2 G(x-y) = \delta^2(x-y)$$

$$G(x-y) = \frac{\int d^D p \ e^{i p \cdot (x-y)}}{(2\pi)^D} \frac{p^2}{p^2}$$

$$= \frac{\Gamma(D/2-1)}{\pi^{D/2} |x-y|^{D-2}}$$

$$G(0) = \lim_{a \to 0} G(a)$$

$$\Rightarrow \ G(x-y) - G(0) = \frac{1}{2\pi} \ln \left( \frac{a}{|x-y|} \right)$$

(D → 2)

$$E = \frac{\int d^2 x \ \omega(x) \ \varphi(x)}{2}$$

$$= \frac{\int d^2 x \ \int d^2 y \ \omega(x) \ G(x-y) \ \omega(y)}{2}$$

$$= \frac{\Phi_0^2}{2} \sum_{i,j} n_i \cdot n_j \ \cdot G(x_i - x_j) \cdot 4\pi^2$$

$$= \pi^2 \Phi_0^2 \left( \sum_{i} n_i \right) G(0) + \frac{\pi^2}{2} \sum_{i,j} n_i \cdot n_j$$

$$\Rightarrow \ G(x_i - x_j) = G(0)$$
\[ \Rightarrow \text{ since } G(0) \to \infty \text{ as } L \to \infty \]

\[ \Rightarrow \text{ only configs with total zero vorticity survive} \]

\[ E = \int \int \phi_0^2 \sum_{i,j} \ln \frac{a}{|x_i - x_j|} \]

\[ \Rightarrow 2D \text{ Coulomb sees!} \]

Energy of a vortex-antivortex pair at distance \( R \) is \[ \frac{\phi_0^2 \ln R/a}{2\pi} \quad (n = \pm 1) \]

Free energy?

\[ F = U - TS \]

\[ \uparrow \quad \uparrow \]

energy entropy

\[ F_{\text{vortex}} = \frac{k_B T}{2} \ln \left( \frac{L}{a} \right) - T \ln \left( \frac{L}{a} \right)^2 \]

\[ \# \text{ of locations} \]

\[ F_{\text{vortex}} < 0 \Rightarrow \text{ vortices proliferate} \]