$$Z = \int \mathcal{D}\chi \mathcal{D}\phi \mathcal{D}c^{+}\mathcal{D}c \; e^{iS} \prod_{x} e^{-i\frac{N}{2}\oint dt \; \varphi_{ab}(x,t)}$$

S is a bilinear form in fermions ⇒ we can integrate them out

⇒ we get an effective action $S_{eff}(\varphi, \chi)$

~~warning: this approach works~~

$$\Rightarrow S_{eff}(\varphi,\chi) \equiv N\,\tilde{S}(\varphi,\chi)$$

$$\begin{aligned}\tilde{S}(\varphi,\chi) = -i\,\mathrm{tr}\ln\Big[&(i\partial_t+\mu)\delta_{ab} + \varphi_{ab}(x,t)\,\delta_{x,x'}\,\delta(t-t') \\ &+ \chi_j^{ab}(x,t)\,\delta_{x',x+e_j} \\ &+ \chi_j^{ba}(x-e_j,t)^{*}\,\delta_{x',x-e_j}\,\delta(t-t')\Big] \\ &- \int dt \sum_j \frac{1}{J}\,|\chi_j^{ab}(x,t)|^2\end{aligned}$$

also we can write

$$\chi_j^{a,b}(x,t) = \underbrace{\rho_j(x,t)}_{\text{pos. def real symmetric}}\; e^{i\overbrace{A_j^{ab}(x,t)}^{\text{in the Lie algebra of } SU(n_c)}}$$

~~Let~~ we will work with fermions in the fund. rep

$\Rightarrow n_c = 1 \Rightarrow$ ~~A_j~~ $a,b=1$

In the large-N limit $N \to \infty$

the path integral can be evaluated in the saddle-point approximation.

Now $\rho_j(x,t)$, $A_\mu(x,t)$ ← abelian.

Saddle-Point Approx.:

$$\frac{\delta \bar{S}_{tot}}{\delta \rho_j(x,t)} = 0 \quad , \quad \frac{\delta \bar{S}_{tot}}{\delta A_\mu(x,t)} = 0$$

$(A_0 \equiv \phi)$

$$S_{tot} = S_{eff} - \frac{1}{2} \sum_x \oint dt \, A_0 \equiv S_{eff} - \sum_x \oint dt \, J_\mu A^\mu$$

$$J_\mu = \frac{1}{2} \delta_{\mu,0}$$

fermion current

$$\Rightarrow \frac{\delta S_{tot}}{\delta A_\mu(x,t)} = \frac{\delta S_{eff}}{\delta A_\mu(x,t)} - J_\mu(x) = j^F_\mu(x,t) - J_\mu(x,t)$$

single occupancy!

since $J_\mu = \frac{1}{2} \delta_{\mu,0} \Rightarrow j^F_0(x,t) = \frac{1}{2}$ (every where)

$$\vec{j}^F(x,t) = 0$$

Two types of solutions

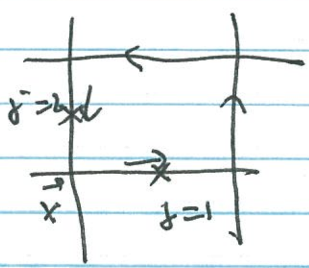
1. Uniform: flux phases
2. Valence-bond crystals (~~been~~ not uniform)

Let's discuss first the uniform solutions.

Case 1 $\Rightarrow$ $\bar{\rho}_j(x,t) \equiv \bar{\rho}$ for all bonds (x,j)

we can have solutions with $A_0 = 0$

but with not trivial $\bar{A}_j$.



$\sum_{\text{plaquette}} \bar{A}_j(x,t) = \bar{B}$ uniform

breaks time-reversal $\bar{B} \leftrightarrow -\bar{B}$

Also $A_j \equiv A_j + 2\pi n_j$ (periodic)

$\Rightarrow \bar{B}$ is defined mod 2π

Time-reversal invariance $\Rightarrow \bar{B} = 0, \pi$ only

Cases with $\bar{B} \neq 0, \pi$ are chiral spin liquids.

Case 2 : $\bar{\rho}_j(x,t) = \bar{\rho}$ or 0 on a set of bonds

$\Rightarrow$ dimer configurations. $\Rightarrow$ break lattice symmetries

but if these symmetries are restored $\Rightarrow \mathbb{Z}_2$ spin liquid

Let's construct the states with $n_c = 1$, $\bar{A}_0 = 0$ and $\bar{\rho}_j$, $\bar{B}$ constant $\Rightarrow$ mean-field theory

$$\Rightarrow H_{MF} = -\sum_{x,j} \bar{\rho}_j(x) \left(c_\alpha^\dagger(x)\, e^{iA_j(x)}\, c_\alpha(x+e_j) + c_\alpha^\dagger(x+e_j)\, e^{-iA_j(x)}\, c_\alpha(x) \right)$$

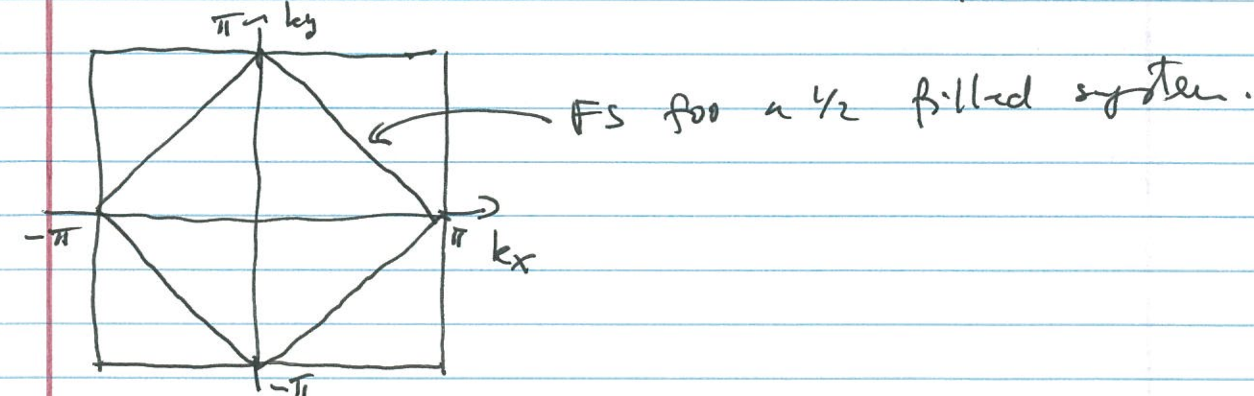
$$+ \frac{N}{J} \sum_{x,j} \bar{\rho}_j^2(x)$$

(I) BZA Phases (Baskaran, Zou, Anderson)

$\bar{\rho}_j(x) \equiv \bar{\rho}$, $\bar{B} = 0$ (no flux)

$\Rightarrow$ we can set $A_j = 0$

$\Rightarrow$ Fermions with a Fermi Surface



$$E_{BZA} = \frac{2NL^2}{J} \bar{\rho}^2 - \frac{8}{\pi^2} NL^2 \bar{\rho}$$

has a minimum if $\bar{\rho} = \frac{2J}{\pi^2}$, $E_{BZA} = -\frac{8NL^2 J}{\pi^4}$

This F.S. is nested and fluctuations in ρ_j can trigger a transition to, e.g., a VB crystal

(II) Flux Phases

$\bar\rho_j(x) \equiv \bar\rho$ (uniform) and $\cancel{B}\ \bar B = \pi$ (uniform)

s.t. $\underset{\text{plaquette}}{\sum} \bar A_j = \pi$ everywhere $\Rightarrow$ A_j is <u>not</u> uniform.

$$H_{flux} = -\bar\rho \sum_{x,j} \left(c^\dagger_\alpha(x) e^{i\bar A_j(x)} c_\alpha(x+e_j) + h.c. \right)$$

$$+ \frac{2NL^2}{J} \bar\rho^2$$

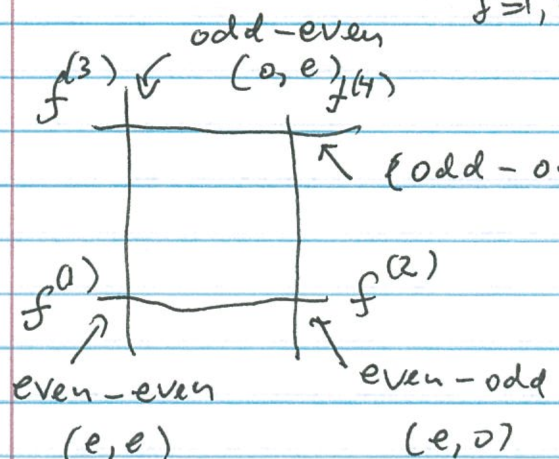
choose $\bar A_j$ / $\bar A_1(x) = +\frac{\pi}{2}$, $\bar A_2(x) = -\frac{\pi}{2}(-1)^{x_1}$

Same as the problem of fermions in a $1/2$ flux quantum

<u>Eq. of motion</u>

$$i\partial_t c_\alpha(x,t) = [c_\alpha(x,t), H_{flux}]$$

$$= -\bar\rho \sum_{j=1,2} \left(e^{i\bar A_j(x)} c_\alpha(x+e_j,\cancel{t}) + e^{-i\bar A_j(x-e_j)} c_\alpha(x-e_j,t) \right)$$



$$\Delta_j \phi(x,t) \equiv \phi(x+e_j,t) - \phi(x-e_j,t)$$

$$i\,\partial_t f^{(a)}_\alpha(x,t) = -i\bar{\rho}\, M^{ab} f^{(b)}_\alpha(x,t)$$

$$\cancel{M^{a,b}} \quad M^{ab} = \begin{pmatrix} 0 & \Delta_1 & -\Delta_2 & 0 \\ \Delta_1 & 0 & 0 & \Delta_2 \\ -\Delta_2 & 0 & 0 & \Delta_1 \\ 0 & \Delta_2 & \Delta_1 & 0 \end{pmatrix}$$

$$\begin{cases} u^{(1)}_\alpha(x,t) \equiv f^{(1)}_\alpha(x,t) + f^{(2)}_\alpha(x+e_1,t) \\ u^{(2)}_\alpha(x,t) \equiv f^{(3)}_\alpha(x+e_2,t) - f^{(4)}_\alpha(x+e_1+e_2,t) \end{cases}$$

$$\begin{cases} v^{(1)}_\alpha(x,t) \equiv f^{(3)}_\alpha(x+e_2,t) + f^{(4)}_\alpha(x+e_1+e_2,t) \\ v^{(2)}_\alpha(x,t) \equiv f^{(1)}_\alpha(x,t) - f^{(2)}_\alpha(x+e_1,t) \end{cases}$$

$$\Rightarrow \quad i\,\partial_t u^{(a)}_\alpha(x,t) = -i\bar{\rho}\,(\sigma_3)_{ab}\,\Delta_1 u^{(b)}_\alpha(x,t) + i\bar{\rho}\,(\sigma_1)_{ab}\,\Delta_2 u^{(b)}_\alpha(x,t)$$

same with $v^{(a)}_\alpha(x,t)$

Def $\quad \gamma_0 = -\sigma_2, \quad \gamma_1 = -i\sigma_1, \quad \gamma_2 = -i\sigma_3$

$$\Rightarrow \quad i(\gamma_0\partial_0 - v_F\,\vec{\gamma}\cdot\vec{\Delta})_{ab}\, u^{(b)}_\alpha = 0$$

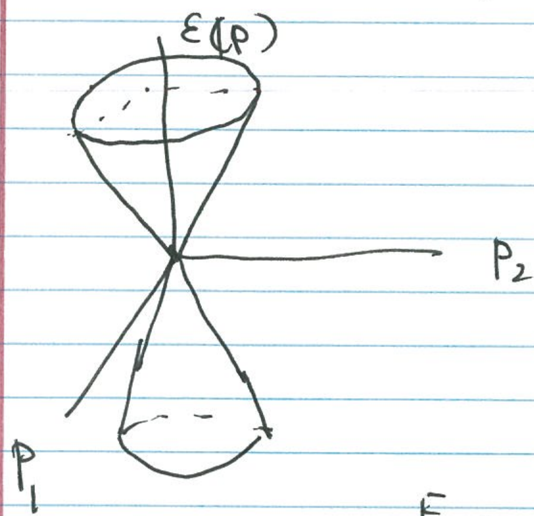
$$i(\gamma_0\partial_0 - v_F\,\vec{\gamma}\cdot\vec{\Delta})_{ab}\, v^{(b)}_\alpha = 0$$

$v_F = 2a\bar{\rho}$

<u>Spectrum</u> $\quad \varepsilon(\vec{p}) = \pm \cancel{2}\bar{\rho}\sqrt{\sin^2 p_1 + \sin^2 p_2}$

$$|p_1| \le \pi/2, \quad |p_2| \le \pi/2$$

$\Rightarrow$ two species of massless <u>Dirac fermions</u>



$$E_{flux} = \frac{2NL^2}{J}\bar{\rho}^2 - 2(2NL^2)\bar{\rho}\int_{|p_i|\le\pi/2} \frac{d^2p}{(2\pi)^2}\sqrt{\sin^2 p_1 + \sin^2 p_2}$$

$$E_{flux} = +\frac{2NL^2}{J}\bar{\rho}^2 - NL^2\bar{\rho}\,\alpha$$

$$\alpha = 4\int_{|\vec{p}_i|\le\pi/2} \frac{d^2p}{(2\pi)^2}\sqrt{\sin^2 p_1 + \sin^2 p_2}$$

Minimize w.r.t. $\bar{\rho} \equiv \frac{1}{4}\alpha J$

$$E_{flux} = -\frac{\alpha^2}{8}NL^2J \approx -0.115\, NL^2 J < E_{BZA}$$

Dimerized Phases

Break translation invariance.

$$\bar{\rho}_j(x) = \begin{cases} \bar{\rho} & \text{if } (x,j) \text{ occupied by a dimer} \\ 0 & \text{otherwise} \end{cases}$$

Huge degeneracy.

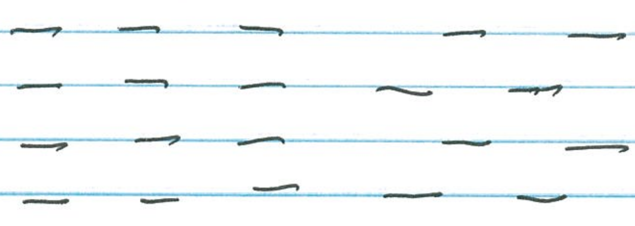
$$E_{VB} = \frac{2NL^2}{J}\bar{\rho}^2 - NL^2\bar{\rho}$$

$$\Rightarrow \bar{\rho} = J/4$$

$$E_{dimer} = -\frac{J}{8} N L^2 < E_{flux}, E_{BZA}$$

Fluctuations ~~bate~~ lifts the degeneracy

$\Rightarrow$ VB crystals



etc.

columnar staggered

Problem

~~the large N~~ the large N solutions are not gauge invariant $\Rightarrow$ fluctuations must restore the local symmetry

Also, the gauge fields are compact

$\Rightarrow$ have magnetic monopole configs.

which may lead to confinement

$\Rightarrow$ Spinons are bound inside electrons.

Other Options

(1) $\Rightarrow$ break $U(1) \to \mathbb{Z}_2$ by a pair field

$\Rightarrow$ $\mathbb{Z}_2$ may be deconfined and topological

($\mathbb{Z}_2$ spin liquid)

(2) break time reversal and suppress

monopoles $\Rightarrow$ Chiral Spin Liquid

(3) Quantum Dimer Phases (VBC's)

(break translation symmetry)