Jordan–Wigner Transformation

A naive look at the Hamiltonian leaves us with a puzzle. We have been raised to think that quadratic Hamiltonians are trivial. So, why the fuzz? Isn't $\hat{H}$ bilinear in $\sigma$'s? The problem is that, in spite of the fact that $\hat{H}$ is a bilinear form in $\sigma$'s, it is not a free theory. It is trivial to check that the equation of motion for $\hat{\sigma}_3(r)$ is not linear since

$$i \partial_t \hat{\sigma}_3(r) = \left[ \hat{\sigma}_3(r), \hat{H} \right] = -2i \hat{\sigma}_2(r) = -2 \hat{\sigma}_3(r) \hat{\sigma}_1(r)$$

and

$$i \partial_t \hat{\sigma}_1(r) = \left[ \hat{\sigma}_1(r), \hat{H} \right] = +\lambda \sum_{j=1}^{N} \left( \hat{\sigma}_3(r+j \cdot \hat{a}) + \hat{\sigma}_3(r-j \cdot \hat{a}) \right)$$
The reason behind this problem is the fact that (equal-time) the commutation relations

\[
[\hat{\sigma}_3(R), \hat{\sigma}_3(R')] = [\hat{\sigma}_3(R), \hat{\sigma}_1(R')] = 0
\]

\[
[\hat{\sigma}_3(R), \hat{\sigma}_1(R')] = 0 \quad (R \neq R')
\]

\[
\{\hat{\sigma}_3(R), \hat{\sigma}_3(R')\} = 0
\]

(and \(\hat{\sigma}_3^2 = \hat{\sigma}_1^2 = I\))

which are not canonical. They seem to describe objects which are bosons, but on different sites, but fermions on the same site. Alternatively they can be regarded as bosons with hard cores. Indeed the raising and lowering operators

\[
\hat{\sigma}^\pm = \frac{1}{2} (\hat{\sigma}_1 \pm i \hat{\sigma}_3)
\]

have the property

\[
\hat{\sigma}^+ |\uparrow\rangle = 0 \quad \hat{\sigma}^+ |\downarrow\rangle = |\uparrow\rangle
\]

\[
\hat{\sigma}^- |\uparrow\rangle = |\downarrow\rangle \quad \hat{\sigma}^- |\downarrow\rangle = 0
\]

and can be regarded as the creation and annihilation operators of some oscillator, but with the hard core constraint that the boson occupation number
\[ \hat{n} = \hat{\sigma}^+ \hat{\sigma}^- \quad \text{only should have eigenvalues 0 and 1} \]

since
\[ \hat{\sigma}^+ |\uparrow\rangle = |\uparrow\rangle \]
\[ \hat{\sigma}^- |\downarrow\rangle = 0 \]

If \( D = 2 \), the quantum problem has \( d = D - 1 = 1 \). It turns out that there is a very neat and useful transformation which will enable us to deal with this problem. This is the Jordan-Wigner transformation. The key idea behind this transformation is that in one-dimensional only, hard-core bosons are equivalent to fermions! Qualitatively this is easy to understand. If the particles are line on a line, the bosons cannot get into exchanged positions by purely dynamical effects since the hard-core condition forbids that possibility. Similarly one-dimensional fermions cannot change their relative ordering as a result of their dynamics (Pauli principle).
Thus, the strategy is to show that our problem is secretly a fermion problem. From now on we will restrict our discussion to $d=1$ ($D=2$).

Let us consider the operator $\hat{K}(n)$

$$\hat{K}(n) = \prod_{j=\frac{n}{2}+1}^{n} (-\hat{\sigma}_z(j))$$

This operator flips all of the spins $\frac{1}{2}$ to the left of site $n+1$.

Clearly on the "high-temperature" ground state $|\uparrow\rangle$

$$\hat{K}(n) |\uparrow\rangle = |\uparrow\rangle$$

(i.e. it is a counting operator in the phase with $\lambda > \lambda_c$)

but, for $\lambda > \lambda_c$, we get

$$\hat{K}(n) |\uparrow\ldots\uparrow\rangle = |\uparrow\ldots\uparrow\rangle$$

this state is a **kink** (or **topological soliton**). Clearly $\hat{K}(n)$ disturbs the boundary conditions (for $\lambda > \lambda_c$) but it does not for $\lambda < \lambda_c$, for where

$$\langle \Psi_0 | \hat{K}(n) | \Psi_0 \rangle \neq 0 \quad (\lambda < \lambda_c)$$

i.e. the high-temperature phase (disordered) is a **understate of kinks**. It is also known as a **disorder operator** (Kadanoff and Ceva).
We will now see that a clever combination of order
(i.e. $\hat{\sigma}_3$) and disorder $\hat{\mathcal{K}}$ operators yields a
fermi field. Let us consider the operators

$$\hat{\chi}_1(j) = \hat{\mathcal{K}}(j-1) \hat{\mathcal{K}}(j) \hat{\sigma}_3(j) \quad (\text{Jordan-Wigner})$$

$$\hat{\chi}_2(j) = i \hat{\mathcal{K}}(j) \hat{\sigma}_3(j)$$
(with $\chi_1(-\frac{1}{2}+1) = \hat{\sigma}_1(-\frac{1}{2}+1)$; $\chi_2 = -\frac{\hat{\sigma}_3}{2}(-\frac{1}{2}+1)$) \n
These operators obey the property $\chi_1 \chi^*_1 = \chi_2 \chi^*_2 = 1 \quad (\forall j)$

since they are products of Pauli matrices, but

$$\{ \chi_1(j), \chi_1(j') \} = 2 \delta_{jj'},$$

$$\{ \chi_1(j), \chi_2(j') \} = \{ \chi_2(j), \chi_2(j') \} = 0 \quad (\text{if } j \neq j')$$

These are almost fermions! Let us define $\hat{\psi}$ and $\hat{\psi}^\dagger$ through

$$\hat{\sigma}^\pm = \frac{1}{2} \left( \hat{\sigma}_3 \mp i \hat{\sigma}_1 \right)$$

$$\hat{\psi}^\dagger(j) = \hat{\mathcal{K}}(j-1) \hat{\sigma}^+(j)$$

$$\hat{\psi}(j) = \hat{\mathcal{K}}(j-1) \hat{\sigma}^-(j)$$

(i.e. $\hat{\psi} = \frac{1}{2}(\chi_1 + i \chi_2)$, $\hat{\psi}^\dagger = \frac{1}{2}(\chi_1 - i \chi_2)$)

It is straightforward to check that

$$\{ \hat{\psi}(j), \hat{\psi}^\dagger(j') \} = \delta_{jj'} \delta_{jj'} \{ \hat{\psi}(j), \hat{\psi}(j') \} = 0$$

It is also easy to invert the transformation.
Let us observe that the fermion number operator $\hat{\Psi}^+_j \Psi_j$ is

$$\hat{\Psi}^+_j, \Psi_j = \hat{K}^{(j-1)} \hat{\sigma}^+_j \hat{K}^{(j-1)} \sigma^- j$$

since $[\hat{K}^{(j-1)}, \hat{\sigma}^+_j] = 0$ and $\hat{K}^2 = 1$, we get

$$\hat{\Psi}^+_j, \Psi_j = \hat{\sigma}^+_j, \hat{\sigma}^- j = \frac{1}{4} (\hat{\sigma}_3 y_j \hat{\sigma}_3 y_j) (\hat{\sigma}_3 y_j + i \hat{\sigma}_3 y_j) = \frac{1}{2} (1 + \hat{\sigma}_1 y_j)$$

Hence

$$-\hat{\sigma}_1 y_j = -2 \hat{\Psi}^+_j \hat{\Psi}_j + 1 \equiv 1 - 2 \hat{n}_y j$$

Since the operator $\hat{n}_y j$ has eigenvalues $0, 1$, we can also write

$$-\hat{\sigma}_1 y_j = e^{i \pi \hat{n}_y j} = e^{i \pi \hat{\Psi}^+_j \hat{\Psi}_j}$$

Hence, the discrete J–W transformation is

$$\hat{\sigma}^+_j = e^{i \pi} \sum_{k \neq j} \hat{\Psi}^+_k \Psi_k \hat{\Psi}^+_j$$

$$\hat{\sigma}^- j = e^{i \pi} \sum_{k < j} \hat{\Psi}^+_k \Psi_k \hat{\Psi}^+_j$$

and

$$\hat{\sigma}_3 y_j = e^{i \pi} \sum_{k < j} \hat{\Psi}^+_k \Psi_k \hat{\Psi}^+_j \Psi_j \hat{\Psi}^+_j \Psi_j$$

$$\hat{\sigma}_2 y_j = e^{i \pi} \sum_{k < j} \hat{\Psi}^+_k \Psi_k \Psi_k \hat{\Psi}^+_j \Psi_j \hat{\Psi}^+_j \Psi_j + \frac{1}{i} (-\hat{\Psi}^+_j \Psi_j + \Psi_j \hat{\Psi}^+_j)$$

Boundary Conditions: If $\hat{\sigma}_3 (\frac{1}{2} + 1) = \gamma \hat{\sigma}_3 (\frac{1}{2} + 1)$ ($\gamma = \pm 1$)

$$\Rightarrow \hat{\Psi} (\frac{1}{2} + 1) = \gamma \hat{\Psi} (\frac{1}{2} + 1)$$

or $\hat{\Psi} (\frac{1}{2} + 1) = \hat{\Psi} (-\frac{1}{2} + 1)$ (where $[Q, H] = 0$)
We can use these results to find a simple form of the Hamiltonian for the equivalent fermi system: (Leven)

\[ \hat{H} = -N_\delta + \sum_{j=-\frac{L}{2}+1} \left( \hat{\Psi}^\dagger_j \hat{\Psi}_j + \lambda \sum_{j=-\frac{L}{2}+1} \left( \hat{\Psi}^\dagger_j \Psi_j(\Psi^\dagger_j + \Psi_j) \right) \right) \]

+ boundary term

Here \( N_\delta = L \). The boundary term is given by

\[ -\lambda \eta \hat{\sigma}_3 \left( \frac{L}{2} \right) \hat{\sigma}_3 \left( -\frac{L}{2} + 1 \right) = -\lambda \eta \hat{\sigma}_3 \left( \Psi^\dagger_{\frac{L}{2}} \Psi_{\frac{L}{2}} \right) + \Psi_{\frac{L}{2} + 1} \Psi_{\frac{L}{2} + 1} \]

where

\[ \hat{\sigma}_3 \hat{\sigma}_3 = \frac{1}{11} \left( -\hat{\sigma}_3 \hat{\sigma}_3 \right) = e^{i\pi N} \]

and \( N \) is the total \# of fermions. Notice that \( \hat{N} \) does not commute with \( \hat{H} \) but \( [e^{i\pi N}, \hat{H}] = 0 \).

Thus we can only tell if the fermions \# is \# even or odd. Also notice that PBC's for the spins (\( \eta = \pm 1 \)) implies that the fermions obey

\[ \hat{\Psi}^\dagger_{\left( \frac{L}{2} + 1 \right)} = \hat{\sigma}_3 \hat{\Psi}_{\left( \frac{L}{2} + 1 \right)} \]

Thus for \( N \) even, the fermions obey PBC's but for \( N \) odd, they obey a PBC's.

In practice, it turns out that \( E_0^- > E_0^+ \) so it can work in the even sector.
**Diagonalization**

The fermion Hamiltonian is bilinear in fermion fields. Thus, it should be diagonalizable by a suitable canonical transformation. Since fermion number is not conserved, this transformation is not just a Fourier transform.

In the even sector

\[ \hat{\Psi} | \Psi \rangle = | \Psi \rangle \]

the F.T. is

\[ (L \text{-even}) \]

\[ \psi(j) = \frac{1}{L} \sum_{k = -\frac{L}{2}}^{\frac{L}{2} - 1} e^{i \frac{2 \pi j k}{L}} \tilde{a}(k) \]

\[ \tilde{a}(k) = \sum_{j = -\frac{L}{2}}^{\frac{L}{2} - 1} e^{-i \frac{2 \pi j k}{L}} \psi(j) \]

such that

\[ \{ \tilde{a}(k), \tilde{a}^\dagger(k') \} = L \delta_{kk'} \]

\[ \{ \tilde{a}(k), \tilde{a}(k') \} = \delta \tilde{a}^\dagger(k), \tilde{a}^\dagger(k') = 0 \]

Also

\[ \frac{1}{L} \sum_{k = -\frac{L}{2}}^{\frac{L}{2} - 1} e^{i \frac{2 \pi j k}{L}} = \delta_{j,0} \]

\[ \frac{1}{L} \sum_{j = -\frac{L}{2} + 1}^{\frac{L}{2} - 1} e^{i \frac{2 \pi j k}{L}} = \delta_{k,0} \]

In the thermodynamic limit \((L \to \infty)\) we get
\[ \lim_{L \to 0} \frac{L}{2\pi} \delta_{k,0} = \delta(k) \quad \text{(Dirac's \ \delta-function)} \]

and \( k \) fills up uniformly the interval \([-\pi, \pi]\)

(this is the first Brillouin zone). Thus

\[ \delta_{j,0} \xrightarrow{L \to \infty} \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-ikj} \quad k = \frac{2\pi}{L} \]

\[ 2\pi \delta(k) = \lim_{L \to \infty} \sum_{j=-\frac{L}{\pi}}^{\frac{L}{\pi}} e^{ikj} \]

Notice that \( 2\pi \delta(0) = L \Rightarrow \delta(0) = \frac{L}{2\pi} \)

Similarly, \( \hat{a}_k = \sum \hat{a}(k) \Rightarrow \{\hat{a}(k), \hat{a}^\dagger(k')\} = 2\pi \delta(k-k') \)

These definitions can be used to get

\[ \sum_{j=-\frac{L}{\pi}}^{\frac{L}{\pi}} \Psi^\dagger(j) \Psi(j) = \frac{L}{\xi^2} \sum_{k,k'} e^{-i\pi \delta(k-k')} \sum_{j=-\frac{L}{\pi}}^{\frac{L}{\pi}} e^{ikj} \]

\[ = \frac{L^2}{\xi^2} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \int_{-\pi}^{\pi} \frac{dk'}{2\pi} \delta(k-k') \hat{a}^\dagger(k) \hat{a}(k') \]

\[ = \frac{L^2}{\xi^2} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \hat{a}^\dagger(k) \hat{a}(k) \]

\[ \sum_{j=-\frac{L}{\pi}}^{\frac{L}{\pi}} \Psi^\dagger(j) \Psi(j) = \frac{L}{\xi} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \int_{-\pi}^{\pi} e^{i\pi \delta(k-k')} \hat{a}^\dagger(k) \hat{a}(k') \]

\[ = \frac{L}{\xi} \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{i\pi \delta(k-k')} \hat{a}^\dagger(k) \hat{a}(k) \]
\[
\lim_{L \to \infty} \frac{L}{2} \sum_{j = -\frac{L}{2} + 1}^{\frac{L}{2}} \psi(j) \psi(j+1) = \sum \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} 2\pi \delta(k+k') e^{-i k'} \hat{a}^\dagger(k) \hat{a}^\dagger(k')
\]
\[
= \sum \int \frac{dk}{2\pi} e^{i k} \hat{a}^\dagger(k) \hat{a}^\dagger(-k)
\]

and

\[
\lim_{L \to \infty} \frac{L}{2} \sum_{j = -\frac{L}{2} + 1}^{\frac{L}{2}} \psi(j) \psi(j+1) = \sum \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} 2\pi \delta(k+k') e^{i k'} \hat{a}(k) \hat{a}(k')
\]
\[
= \sum \int \frac{dk}{2\pi} e^{-i k} \hat{a}(k) \hat{a}(-k)
\]

By collecting terms, we have

\[
H = -L + 2 \sum \int \frac{dk}{2\pi} (1 + \lambda \cos k) \hat{a}^\dagger(k) \hat{a}(k)
\]
\[
+ \lambda \sum \int \frac{dk}{2\pi} (e^{i k} \hat{a}^\dagger(k) \hat{a}^\dagger(-k) - e^{-i k} \hat{a}(k) \hat{a}(-k))
\]

This Hamiltonian is reminiscent of the pairing Hamiltonian of the BCS theory of superconductivity. Notice that we can rewrite \( H \) in the form

\[
H = -L + 2 \sum \int \frac{dk}{2\pi} (1 + \lambda \cos k) \left( \hat{a}^\dagger(k) \hat{a}(k) + \hat{a}^\dagger(-k) \hat{a}(-k) \right)
\]
\[ + \frac{1}{2\pi} \int_0^\pi \text{d}k \, 2i \mathbf{a} \sin k \, (a^\dagger(k) a^\dagger(-k) + a(k) a(-k)) \]

It is possible to write \( H \) in terms of the spinor field \( \hat{\Psi}(k) \)

\[ \hat{\Psi}(k) = \begin{pmatrix} \hat{a}^\dagger(k) \\ \hat{a}(-k) \end{pmatrix} \]

\[ \hat{\Psi}^\dagger(k) = \begin{pmatrix} \hat{a}(k) & \hat{a}^\dagger(-k) \end{pmatrix} \]

Notice that the two components of \( \hat{\Psi} \) are not independent. Indeed, if we denote by \( \hat{C} \) the matrix

\[ \hat{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

we get

\[ \hat{\Psi}^\dagger(k) = \left[ \hat{C} \hat{\Psi}(-k) \right]^T = \hat{\Psi}^T(-k) \hat{C} \]

This is a real spinor field (or Majorana fermion) (not Dirac which is complex). In terms of \( \hat{\Psi}(k) \)

we can write \( H \)

\[ H = -L \left( -1 + \frac{1}{2\pi} \int_0^\pi \text{d}k \, 2(1 + \lambda \cos k) \right) - \frac{1}{2\pi} \int_0^\pi \text{d}k \, \hat{\Psi}^\dagger(k) \begin{pmatrix} 3(1 + \sin k) & (2i \sin k) \\ (2i \sin k) & (2i \sin k) \end{pmatrix} \hat{\Psi}(k) \]
Rogolinov Transformation:

\[ a(k) = u(k) \gamma(k) - i v(k) \gamma^\dagger(-k) \]
\[ a(-k) = u(k) \gamma(-k) + i v(k) \gamma^\dagger(k) \]

with \( u(k) \) and \( v(k) \) real functions of \( k \) (to be determined below)

and

\[ \gamma(k) = u(k) a(k) + i v(k) a^\dagger(-k) \]
\[ \gamma(-k) = u(k) a(-k) - i v(k) a^\dagger(k) \]

This transformation is commutative, i.e.

\[ \{ a(k), a(k')^\dagger \} = 2\pi \delta(k-k') \Rightarrow \{ \gamma(k), \gamma(k')^\dagger \} = 2\pi \delta(k-k') \]

iff

\[ u(k)^2 + v(k)^2 = 1 \]

i.e.

\[ u(k) = \cos \Theta(k) \]
\[ v(k) = \sin \Theta(k) \]

We will determine \( \Theta(k) \) and \( \Theta(-k) \) by demanding that the formation-inversion term cancel out (in terms of the \( \gamma \)'s).
\[ a^+(k) a(k) + a^+(-k) a(-k) = \]
\[ = \alpha^2(k) \left( \eta^+(k) \eta(k) + \eta^+(-k) \eta(-k) \right) + \]
\[ + u^2(k) \left( \eta(-k) \eta^+(-k) + \eta(k) \eta^+(k) \right) + \]
\[ + i \cdot u(k) v(k) \left[ - \eta^+(k) \eta^+(-k) + \eta(k) \eta(-k) + \eta^+(k) \eta(k) - \eta^+(-k) \eta(-k) \right] \]
\[ a^+(k) a^+(-k) - a(-k) a(k) = \]
\[ = \alpha^2(k) \left[ \eta^+(k) \eta(k) + \eta(-k) \eta(-k) \right] + \]
\[ + u^2(k) \left[ \eta(-k) \eta(k) + \eta^+(-k) \eta^+(k) \right] + \]
\[ + i \cdot u(k) v(k) \left[ - \eta^+(k) \eta^+(-k) + \eta^+(k) \eta(k) + \eta(k) \eta^+(k) - \eta^+(-k) \eta(-k) \right] \]

with
\[ \alpha(k) = 2 \left( 1 + \lambda \cos k \right) \]
\[ \beta(k) = 2 \lambda \sin k \]

Hence
\[ \alpha(k) \left( a^+(k) a(k) + a^+(-k) a(-k) \right) + i \beta(k) \left[ a^+(k) a^+(-k) - a(-k) a(k) \right] = \]
\[ = \left[ \alpha(k) \left( u^2(k) - v^2(k) \right) + 2 \beta(k) \left( u(k) v(k) \right) \right] \left( \eta^+(k) \eta(k) + \eta^+(-k) \eta(-k) \right) + \]
\[ + \left[ -2 u(k) v(k) \alpha(k) + \beta(k) \left( u^2(k) - v^2(k) \right) \right] i \left( \eta^+(k) \eta^+(-k) + \eta(k) \eta(-k) \right) \]
\[ + 2 \left( u^2(k) \alpha(k) - u(k) v(k) \beta(k) \right) \]
The condition is:

\[-2\alpha(k)u(k)v(k) + \beta(k) \left((u^2(k) - v^2(k))\right) = 0\]

\[u(k) = \sin^2\theta(k) \quad \text{and} \quad v(k) = \sin^2\theta(k) \Rightarrow\]

\[-\alpha(k) \sin 2\theta(k) + \beta(k) \cos 2\theta(k) = 0\]

and

\[\tan 2\theta(k) = \frac{\beta(k)}{\alpha(k)} = \infty\]

\[\Rightarrow \tan (2\theta(k)) = \frac{\lambda \sin k}{1 + \lambda \sin k}\]

With this choice, \(H\) becomes

\[H = \int_0^{\pi} \omega(k) \left(\gamma(k) \gamma(k) + \gamma(-k) \gamma(-k)\right) + \varepsilon_0 L\]

where

\[\varepsilon_0(\lambda) = -1 + 2 \int_0^{\pi} \frac{dk}{2\pi} \left(\omega^2(k) 2\lambda \sin k - \omega(k) \omega(-k) 2(1 + \lambda \cos k)\right)\]

\[= -1 + \theta \int_0^{\pi} \frac{dk}{2\pi} \left[4\lambda \sin k \sin^2\theta(k) - 2 \sin 2\theta(k)(1 + \lambda \cos k)\right]\]

\[\omega(k) = \alpha(k) \cos 2\theta(k) + \beta(k) \sin 2\theta(k)\]

\[= 2 \left[(1 + \lambda \cos k) \cos 2\theta(k) + \theta \lambda \sin k \sin \theta(k)\right]\]
I will choose the signs of $\cos 2\theta(k)$ and $\sin 2\theta(k)$ in such a way that $\omega(k) \geq 0$.

$$\omega(k) = \alpha(k) \cos 2\theta(k) + \beta(k) \sin 2\theta(k)$$

choose

$\text{sign} \cos 2\theta(k) = \text{sign} \alpha(k)$

$\text{sign} \sin 2\theta(k) = \text{sign} \beta(k)$

$$\omega(k) = \left| \alpha(k) \right| |\cos 2\theta(k)| + \left| \beta(k) \right| |\sin 2\theta(k)|$$

But

$$|\cos 2\theta(k)| = \frac{1}{\sqrt{1 + \tan^2 2\theta(k)}} = \frac{1}{\sqrt{1 + \left(\frac{\beta(k)}{\alpha(k)}\right)^2}}$$

$$|\sin 2\theta(k)| = \frac{\left| \tan 2\theta(k) \right|}{\sqrt{1 + \tan^2 2\theta(k)}} = \frac{\left| \frac{\beta(k)}{\alpha(k)} \right|}{\sqrt{1 + \left(\frac{\beta(k)}{\alpha(k)}\right)^2}}$$

$$\Rightarrow \omega(k) = \frac{|\alpha(k)|}{\sqrt{1 + \left(\frac{\beta(k)}{\alpha(k)}\right)^2}} + \frac{\left| \beta(k) \right| |\frac{\beta(k)}{\alpha(k)}|}{\sqrt{1 + \left(\frac{\beta(k)}{\alpha(k)}\right)^2}} = \frac{|\alpha(k)| \left(1 + \left(\frac{\beta(k)}{\alpha(k)}\right)^2\right)}{\sqrt{1 + \left(\frac{\beta(k)}{\alpha(k)}\right)^2}}$$

$$\Rightarrow \omega(k) = |\alpha(k)| \sqrt{1 + \left(\frac{\beta(k)}{\alpha(k)}\right)^2} \Rightarrow \omega(k) = \sqrt{\left(\frac{\left| \alpha(k) \right|}{\sqrt{1 + \left(\frac{\beta(k)}{\alpha(k)}\right)^2}}\right)^2 + \left(\frac{\left| \beta(k) \right|}{\sqrt{1 + \left(\frac{\beta(k)}{\alpha(k)}\right)^2}}\right)^2}$$

$$\Rightarrow \omega(k) = \sqrt{\left(1 + \lambda \omega k\right)^2 + \lambda^2 \sin^2 k}$$
After some algebra we get

\[ \omega(k) = 2\sqrt{1 + \lambda^2 + 2\lambda \cos k} \]

which is clearly positive. With these choices, the ground state \( |0\rangle \) is simply the state annihilated by all the \( \hat{a} \) destruction operators.

\[ \eta(k) |0\rangle = 0 \]

\[ \eta(-k) |0\rangle = 0 \]

The ground state energy (density) is thus \( E_0 \).

\[ E_0(\lambda) = 1 + \int_0^\pi \frac{dk}{2\pi} \left[ 4\lambda \sin k \sin^2 \theta(k) - 2\sin 2\theta(k) \right] (1 + \lambda \cos k) \]

\[ \Rightarrow E_0(\lambda) = -1 - \int_0^\pi \frac{dk}{2\pi} \omega(k) = -\frac{1}{2} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \omega(k) < 0 \]

First excited state:

It is a fermion state

\[ |k\rangle \equiv \eta^\dagger(k) |0\rangle \]

\[ H |k\rangle = E_0 (E_0 + \omega(k)) |k\rangle \]

Excitation energy \( E(k) = \omega(k) > 0 \)

Energy gap \( \Theta(\lambda) = \min E(k) \).
\[ \omega(k) = 2\sqrt{1 + \lambda^2 + i\lambda \sin k} = 2\sqrt{(1 - \lambda)^2 + 4\lambda^2 \cos^2 \frac{k}{2}} \]

Notice that exactly at \( \lambda = 1 \) the gap goes to zero.

This is the phase transition point.

Indeed

\[ Z = \text{tr} T^N \sim \text{tr} e^{-\beta H} \rightarrow \beta \rightarrow \infty \]

\[ Z = e^{-\beta L f} \]

\[ \Rightarrow f \approx E_0(\lambda) \quad (f = \text{free energy}) \]

(free energy density of the 2D IM)

\[ E_0(\lambda) = -2 \int_0^{\pi} \frac{dk}{2\pi} \sqrt{(1 + \lambda)^2 - 4\lambda \sin^2 \frac{k}{2}} \]

\[ = -\frac{2|1+\lambda|}{\pi} \int_{0}^{\pi/2} dx \sqrt{1 - (-\lambda^2) \sin^2 x} = -\frac{2|1+\lambda|}{\pi} \frac{\text{E}(\frac{\pi}{2}, \sqrt{1-\lambda^2})}{\sqrt{1-\lambda^2}} \]

\[ -\lambda^2 = \frac{4\lambda}{(1+\lambda)^2} \Rightarrow |8| = \left| \frac{1-\lambda}{1+\lambda} \right| \]

\[ \text{elliptic function} \]
If $\lambda \to 1 = \gamma = 0$ and

$$E\left(\frac{\pi}{2}, \sqrt{1-\gamma^2}\right) \approx \frac{1+\gamma^2}{4} \left(\ln \frac{16}{8} - 1\right) + o(\gamma^4)$$

then

$$E_0(\lambda) \approx -2\left(\frac{1+\lambda}{\pi}\right) \left(1 + \frac{1}{4} \left(\ln \frac{16}{8} - 1\right) + \cdots \right)$$

$$= E_0^{\text{sing}} + E_0^{\text{reg}}$$

Singular term $E_0^{\text{sing}} = -\frac{1}{\pi} \left[1 + \frac{(1-\lambda)^2}{16} \left(\ln \left(\frac{6\gamma}{1-\gamma}\right)^2 - 1\right) + \cdots \right]$.

$$t = |1-\lambda|$$

Specific heat:

$$C = C^{\text{sing}} + C^{\text{reg}}$$

$$C^{\text{sing}} = -\frac{\partial^2 E_0}{\partial \ln \gamma^2} + \frac{1}{2\pi} \ln \left(\frac{8}{|t|}\right) - \frac{3}{4} + \cdots$$

This is the famous logarithmic divergence of the specific heat of the 2D Ising model.
Also the gap $G(\mathcal{g}) \approx A |\lambda - \lambda_c|^{\nu_g}$

with $\lambda_c = 1$, $A = 1$ and $\nu = 1$.

$\xi = \text{correlation length}$

$$\xi = \frac{\nu_5}{G}$$

But $\nu_5 (\lambda = 1) = 2 \Rightarrow \xi (\lambda) = \frac{1}{G(\lambda)} = \frac{1}{|\lambda - 1|}$

Correlation length exponent $\nu_5 = +1$

Equations of Motion for $\lambda \approx 1$

The (Majorana) fermions $X_1(j)$ and $X_2(j)$ have simple equations of motion. Indeed, one finds

$$i \partial_t X_1(j) = i X_2(j) - i \lambda X_2(j-1)$$

$$i \partial_t X_2(j) = -i X_1(j) + i \lambda X_1(j+1)$$

Restore a lattice constant $a_0 \neq 1$ and set $x_j = j a_0$.

$\Rightarrow X_2(j-1) \approx X_2(x_j) - a_0 \partial_x X_2(x_j)$

$X_1(j+1) \approx X_1(x_j) + a_0 \partial_x X_1(x_j)$

i.e.

$$\frac{i}{a_0 \lambda} \partial_t X_1 \approx i \left(1 - \frac{\lambda}{a_0 \lambda} \right) X_2 + i \partial_x X_2$$

$$\frac{i}{a_0 \lambda} \partial_t X_2 \approx -i \left(1 - \frac{\lambda}{a_0 \lambda} \right) X_1 + i \partial_x X_1$$

Rescaling time: $t \rightarrow t \lambda (a_0 \lambda)$ we get

$x \rightarrow x_1$
\[ i \partial_0 X_1 - i \partial_1 X_2 + i \left( \frac{1}{\alpha_0 \lambda} \right) X_2 = 0 \]

\[ i \partial_0 X_2 - i \partial_1 X_1 - i \left( \frac{1}{\alpha_0 \lambda} \right) X_1 = 0 \]

If we define \( m = \lim_{\lambda \to 1} \left( \frac{1 - \lambda}{\alpha_0 \lambda} \right) \) (Scaling limit)

we obtain

\[ i \partial_0 X_1 - i \partial_1 X_2 + i m X_2 = 0 \]

\[ i \partial_0 X_2 - i \partial_1 X_1 - i m X_1 = 0 \]

With the notation: \( \gamma_0 = -\sigma_2 \quad \gamma_1 = i \sigma_3 \quad \gamma_5 = \sigma_3 \)

we found that the spinor \( (X_1 \quad X_2) \) obeys the 1+1-dimensional Dirac equation

\[ (i \partial - m) X = 0 \]

Notice that \( X \) is a real field (Majorana!)

\[ \Rightarrow X^+ = X^\dagger \]

Thus, the scaling limit of \( \lambda \to 1 \) (i.e., getting asymptotically close to the phase transition) and \( \alpha_0 \to 0 \) ("continuum limit") the 2D IM is equivalent (or defined) to the field theory of free Majorana fermions. Notice that this works only at distances long compared with the lattice constant (i.e., \( \alpha_0 \to 0 \)) but comparable to \( \xi \approx \frac{1}{|m|} \).