# 5

# Path Integrals in Quantum Mechanics and Quantum Field Theory

In chapter 4 we discussed the Hilbert space picture of Quantum Mechanics and Quantum Field Theory for the case of a free relativistic scalar fields. Here we will present the *Path Integral* picture of Quantum Mechanics and of relativistic scalar field theories.

The Path Integral picture is important for two reasons. First, it offers an alternative, complementary, picture of Quantum Mechanics in which the role of the classical limit is apparent. Secondly, it gives a direct route to the study regimes where perturbation theory is either inadequate or fails completely. In Quantum mechanics a standard approach to such problems is the WKB approximation, of Wentzel, Kramers and Brillouin. However, as it happens, it is extremely difficult (if not impossible) the generalize the WKB approximation to a Quantum Field Theory. Instead, the non-perturbative treatment of the Feynman path integral, which in Quantum Mechanics is equivalent to WKB, is generalizable to non-perturbative problems in Quantum Field Theory. In this chapter we will use path integrals only for bosonic systems, such as scalar fields. In subsequent chapters we will also give a full treatment of the path integral, including its applications to fermionic fields, abelian and non-abelian gauge fields, classical statistical mechanics, and non-relativistic many body systems.

There is a huge literature on path integrals, going back to the original papers by Dirac (Dirac, 1933), and particularly Feynman's 1942 PhD Thesis (Feynman, 2005), and his review paper (Feynman, 1948). Popular textbooks on path integrals include the classic by Feynman and Hibbs (Feynman and Hibbs, 1965), and Schulman's book (Schulman, 1981), among many others.

## 5.1 Path Integrals and Quantum Mechanics

Consider a simple quantum mechanical system whose dynamics can be described by a generalized coordinate operator $\hat{q}$. We want to compute the amplitude

$$F(q_f, t_f | q_i, t_i) = \langle q_f, t_f | q_i, t_i \rangle \tag{5.1}$$

known as the Wightman function. This function represents the amplitude to find the system at coordinate $q_f$ at the final time $t_f$ knowing that it was at coordinate $q_i$ at the initial time $t_i$. The amplitude $F(q_f, t_f | q_i, t_i)$ is just a matrix element of the evolution operator

$$F(q_f, t_f | q_i, t_i) = \langle q_f | e^{i\hat{H}(t_i - t_f)/\hbar} | q_i \rangle \tag{5.2}$$

Let us set, for simplicity, $|q_i, t_i\rangle = |0, 0\rangle$ and $|q_f, t_f\rangle = |q, t\rangle$. Then, from the definition of this matrix element, we find out that it obeys

$$\lim_{t \to 0} F(q, t | 0, 0) = \langle q | 0 \rangle = \delta(q) \tag{5.3}$$

Furthermore, after some algebra we also find that

$$\begin{aligned} i\hbar \frac{\partial F}{\partial t} =& i\hbar \frac{\partial}{\partial t} \langle q, t | 0, 0 \rangle = i\hbar \frac{\partial}{\partial t} \langle q | e^{-i\hat{H}t/\hbar} | 0 \rangle \\ =& \langle q | \hat{H} e^{-i\hat{H}t/\hbar} | 0 \rangle \\ =& \int dq' \langle q | \hat{H} | q' \rangle \langle q' | e^{-i\hat{H}t/\hbar} | 0 \rangle \end{aligned} \tag{5.4}$$

where we have used that, since $\{|q\rangle\}$ is a complete set of states, the identity operator $I$ has the expansion, called the *resolution of the identity*,

$$I = \int dq' |q'\rangle\langle q'| \tag{5.5}$$

Here we have assumed that the states are orthonormal,

$$\langle q | q' \rangle = \delta(q - q') \tag{5.6}$$

Hence,

$$i\hbar \frac{\partial}{\partial t} F(q, t | 0, 0) = \int dq' \langle q | \hat{H} | q' \rangle F(q', t | 0, 0) \equiv \hat{H}_q F(q, t | 0, 0) \tag{5.7}$$

In other words, $F(q, t | 0, 0)$ is the solution of the Schrödinger Equation that satisfies the initial condition of Eq. (5.3). For this reason, the amplitude $F(q, t | 0, 0)$ is called the *Schrödinger Propagator*.

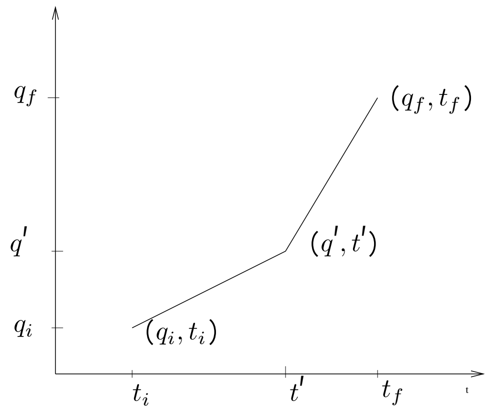


Figure 5.1 The amplitude to go from $|q_i, t_i\rangle$ to $|q_f, t_f\rangle$ is a sum of products of amplitudes through the intermediate states $|q', t'\rangle$.

The superposition principle tells us that the amplitude to find the system in the final state at the final time is the sum of amplitudes of the form

$$F(q_f, t_f | q_i, t_i) = \int dq' \langle q_f, t_f | q', t' \rangle \langle q', t' | q_i, t_i \rangle \tag{5.8}$$

where the system is in an arbitrary set of states at an intermediate time $t'$. Here we represented this situation by inserting the identity operator $I$ at the intermediate time $t'$ in the form of the resolution of the identity of Eq. (5.8).

Let us next define a *partition* of the time interval $[t_i, t_f]$ into $N$ sub-intervals each of length $\Delta t$,

$$t_f - t_i = N \Delta t \tag{5.9}$$

Let $\{t_j\}$, with $j = 0, \ldots, N+1$, denote a set of points in the interval $[t_i, t_f]$, such that

$$t_i = t_0 \leq t_1 \leq \ldots \leq t_N \leq t_{N+1} = t_f \tag{5.10}$$

Clearly, $t_k = t_0 + k\Delta t$, for $k = 1, \ldots, N+1$. By repeating the procedure used in Eq.(5.8) of inserting the resolution of the identity at the intermediate times $\{t_k\}$, we find

$$\begin{aligned} F(q_f, t_f | q_i, t_i) = \int dq_1 \ldots dq_N \langle q_f, t_f | q_N, t_N \rangle \langle q_N, t_N | q_{N-1}, t_{N-1} \rangle \times \ldots \\ \times \ldots \langle q_j, t_j | q_{j-1}, t_{j-1} \rangle \ldots \langle q_1, t_1 | q_i, t_i \rangle \end{aligned} \tag{5.11}$$

Each factor $\langle q_j, t_j | q_{j-1}, t_{j-1} \rangle$ in Eq.(5.11) has the form

$$\langle q_j, t_j | q_{j-1}, t_{j-1} \rangle = \langle q_j | e^{-i\hat{H}(t_j - t_{j-1})/\hbar} | q_{j-1} \rangle \equiv \langle q_j | e^{-i\hat{H}\Delta t/\hbar} | q_{j-1} \rangle \tag{5.12}$$

In the limit $N \to \infty$, with $|t_f - t_i|$ fixed and finite, the interval $\Delta t$ becomes infinitesimally small and $\Delta t \to 0$. Hence, as $N \to \infty$ we can approximate the expression for $\langle q_j, t_j | q_{j-1}, t_{j-1} \rangle$ in Eq.(5.12) as follows

$$\begin{aligned} \langle q_j, t_j | q_{j-1}, t_{j-1} \rangle &= \langle q_j | e^{-iH\Delta t/\hbar} | q_{j-1} \rangle \\ &= \langle q_j | \Big\{ I - i\frac{\Delta t}{\hbar}\hat{H} + O((\Delta t)^2) \Big\} | q_{j-1} \rangle \\ &= \delta(q_j - q_{j-1}) - i\frac{\Delta t}{\hbar}\langle q_j | \hat{H} | q_{j-1} \rangle + O((\Delta t)^2) \end{aligned} \tag{5.13}$$

which becomes asymptotically exact as $N \to \infty$.

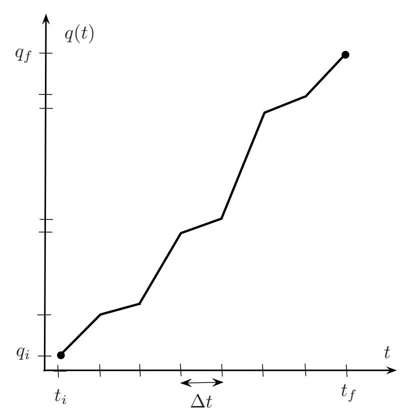


Figure 5.2 A history $q(t)$ of the system.

We can also introduce at each intermediate time $t_j$ a complete set of momentum eigenstates $\{|p\rangle\}$ using their resolution of the identity

$$I = \int_{-\infty}^{\infty} dp\ |p\rangle\langle p| \tag{5.14}$$

Recall that the overlap between the states $|q\rangle$ and $|p\rangle$ is

$$\langle q | p \rangle = \frac{1}{\sqrt{2\pi\hbar}}\ e^{ipq/\hbar} \tag{5.15}$$

For a typical Hamiltonian of the form

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q}) \tag{5.16}$$

its matrix elements are

$$\langle q_j|\hat{H}|q_{j-1}\rangle = \int_{-\infty}^{\infty} \frac{dp_j}{2\pi\hbar}\, e^{ip_j(q_j-q_{j-1})/\hbar} \left[\frac{p_j^2}{2m} + V(q_j)\right] \tag{5.17}$$

Within the same level of approximation we can also write

$$\langle q_j, t_j|q_{j-1}, t_{j-1}\rangle \approx \int \frac{dp_j}{2\pi\hbar} \exp\left[i\left(\frac{p_j}{\hbar}(q_j - q_{j-1}) - \Delta t\, H\left(p_j, \frac{q_j + q_{j-1}}{2}\right)\right)\right] \tag{5.18}$$

where we have introduce the "mid-point rule" which amounts to the replacement $q_j \to \frac{1}{2}\left(q_j + q_{j-1}\right)$ inside the Hamiltonian $H(p, q)$. Putting everything together we find that the matrix element $\langle q_f, t_f|q_i, t_i\rangle$ becomes

$$\begin{aligned}\langle q_f, t_f|q_i, t_i\rangle = \lim_{N\to\infty} \int \prod_{j=1}^{N} dq_j \int_{-\infty}^{\infty} \prod_{j=1}^{N+1} \frac{dp_j}{2\pi\hbar} \\ \exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N+1}\left[p_j(q_j - q_{j-1}) - \Delta t\, H\left(p_j, \frac{q_j + q_{j-1}}{2}\right)\right]\right\}\end{aligned} \tag{5.19}$$

Therefore, in the limit $N \to \infty$, holding $|t_i - t_f|$ fixed, the amplitude $\langle q_f, t_f|q_i, t_i\rangle$ is given by the (formal) expression

$$\langle q_f, t_f|q_i, t_i\rangle = \int \mathcal{D}p\mathcal{D}q\ e^{\displaystyle\frac{i}{\hbar}\int_{t_i}^{t_f} dt\,[p\dot{q} - H(p,q)]} \tag{5.20}$$

where we have used the notation

$$\mathcal{D}p\mathcal{D}q \equiv \lim_{N\to\infty} \prod_{j=1}^{N} \frac{dp_j dq_j}{2\pi\hbar} \tag{5.21}$$

which defines the integration measure. The functions, or *configurations*, $(q(t), p(t))$ must satisfy the initial and final conditions

$$q(t_i) = q_i, \qquad q(t_f) = q_f \tag{5.22}$$

Thus the matrix element $\langle q_f, t_f|q_i, t_i\rangle$ can be expressed as a sum over histories in *phase space*. The weight of each history is the exponential factor of Eq. (5.20). Notice that the quantity in brackets it is just the Lagrangian

$$L = p\dot{q} - H(p, q) \tag{5.23}$$

Thus the matrix element is just

$$\langle q_f, t_f | q, t \rangle = \int \mathcal{D}p\mathcal{D}q \; e^{\frac{i}{\hbar}S(q,p)} \tag{5.24}$$

where $S(q,p)$ is the action of each history $(q(t), p(t))$. Also notice that the sum (or integral) runs over *independent* functions $q(t)$ and $p(t)$ which are not required to satisfy any constraint (apart from the initial and final conditions) and, in particular they are not the solution of the equations of motion. Expressions of these type are known as *path-integrals*. They are also called functional integrals, since the integration measure is a sum over a space of *functions*, instead of a field of numbers as in a conventional integral.

Using a Gaussian integral of the form (which involves an analytic continuation)

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} \; e^{i(p\dot{q} - \frac{p^2}{2m})\frac{\Delta t}{\hbar}} = \sqrt{\frac{m}{2\pi i\hbar\Delta t}} \; e^{i\frac{\Delta t}{2\hbar}\dot{q}^2} \tag{5.25}$$

we can integrate out explicitly the momenta in the path-integral and find a formula that involves only the histories of the coordinate alone. Notice that there are no initial and final conditions on the momenta since the initial and final states have well defined positions. The result is

$$\langle q_f, t_f | q_i, t_i \rangle = \int \mathcal{D}q \;\; e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt \; L(q,\dot{q})} \tag{5.26}$$

which is known as the *Feynman Path Integral* (Feynman, 2005, 1948). Here $L(q,\dot{q})$ is the Lagrangian,

$$L(q,\dot{q}) = \frac{1}{2}m\dot{q}^2 - V(q) \tag{5.27}$$

and the sum over histories $q(t)$ is restricted by the boundary conditions $q(t_i) = q_i$ and $q(t_f) = q_f$.

The Feynman path-integral tells us that in the correspondence limit, $\hbar \to 0$, the only history (or possibly histories) that contribute significantly to the path integral must be those that leave the action $S$ stationary since, otherwise, the contributions of the rapidly oscillating exponential would add up to zero. In other words, in the classical limit there is only one history $q_c(t)$ that contributes. For this history, $q_c(t)$, the action $S$ is stationary, $\delta S = 0$, and $q_c(t)$ is the solution of the Classical Equation of Motion

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 \tag{5.28}$$

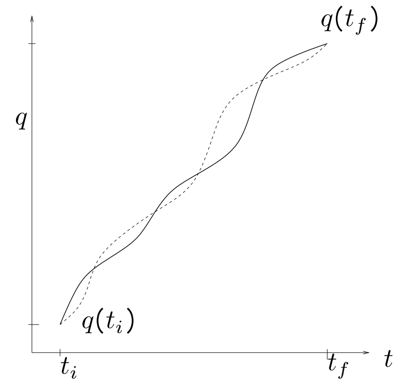


Figure 5.3 Two histories with the same initial and final states.

In other terms, in the correspondence limit $\hbar \to 0$, the evaluation of the Feynman path integral reduces to the requirement that the Least Action Principle should hold. This is the classical limit.

## 5.2 Evaluating path integrals in Quantum Mechanics

Let us first discuss the following problem. We wish to know how to compute the amplitude $\langle q_f, t_f | q_i, t_i \rangle$ for a dynamical system whose Lagrangian has the standard form of Eq. (5.27). For simplicity we will begin with a linear harmonic oscillator.

The Hamiltonian for a linear harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2 \tag{5.29}$$

and the associated Lagrangian is

$$L = \frac{m}{2} \dot{q}^2 - \frac{m\omega^2}{2} q^2 \tag{5.30}$$

Let $q_c(t)$ be the classical trajectory. It is the solution of the classical equations of motion

$$\frac{d^2 q_c}{dt^2} + \omega^2 q_c = 0 \tag{5.31}$$

Let us denote by $q(t)$ an arbitrary history of the system and by $\xi(t)$ its deviation from the classical solution $q_c(t)$. Since all the histories, including the classical trajectory $q_c(t)$, obey the *same* initial and final conditions

$$q(t_i) = q_i \qquad q(t_f) = q_f \tag{5.32}$$

it follows that $\xi(t)$ obeys instead *vanishing* initial and final conditions:

$$\xi(t_i) = \xi(t_f) = 0 \tag{5.33}$$

After some trivial algebra it is easy to show that the action $S$ for an arbitrary history $q(t)$ becomes

$$S(q,\dot{q}) = S(q_c,\dot{q}_c) + S(\xi,\dot{\xi}) + \int_{t_i}^{t_f} dt\, \frac{d}{dt}\left[m\xi\frac{dq_c}{dt}\right] + \int_{t_i}^{t_f} dt\, m\xi\left(\frac{d^2q_c}{dt^2} + \omega^2 q_c\right) \tag{5.34}$$

The third term vanishes due to the boundary conditions obeyed by the fluctuations $\xi(t)$, Eq. (5.33). The last term also vanishes since $q_c$ is a solution of the classical equation of motion Eq. (5.31). These two features hold for all systems, even if they are not harmonic. However, the Lagrangian (and hence the action) for $\xi$, the second term in Eq. (5.34), in general is not the same as the action for the classical trajectory (the first term). Only for the harmonic oscillator $S(\xi,\dot{\xi})$ has the same form as $S(q_c,\dot{q}_c)$.

Hence, for a harmonic oscillator, we get the path integral

$$\langle q_f, t_f | q_i, t_i\rangle = e^{\frac{i}{\hbar}S(q_c,\dot{q}_c)} \int_{\xi(t_i)=\xi(t_f)=0} \mathcal{D}\xi\; e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt L(\xi,\dot{\xi})} \tag{5.35}$$

Notice that the information on the initial and final states enters only through the factor associated with the classical trajectory. For the linear harmonic oscillator, the quantum mechanical contribution is independent of the initial and final states. Thus, we need to do two things: 1) we need an explicit solution $q_c(t)$ of the equation of motion, for which we will compute $S(q_c,\dot{q}_c)$, and 2) we need to compute the quantum mechanical correction, the last factor in Eq. (5.35), which measures the strength of the quantum fluctuations.

For a general dynamical system, whose Lagrangian has the form of Eq. (5.27), the action of Eq. (5.34) takes the form

$$\begin{aligned} S(q,\dot{q}) = {} & S(q_c,\dot{q}_c) + S_{\text{eff}}(\xi,\dot{\xi};q_c) \\ & + \int_{t_i}^{t_f} dt\, \frac{d}{dt}\left[m\xi\frac{dq_c}{dt}\right] + \int_{t_i}^{t_f} dt\, \left(m\frac{d^2q_c}{dt^2} + \left.\frac{\partial V}{\partial q}\right|_{q_c}\right)\xi(t) \end{aligned} \tag{5.36}$$

where the $S_{\text{eff}}$ is the effective action for the fluctuations $\xi(t)$ which has the form

$$S_{\text{eff}}(\xi,\dot{\xi}) = \int_{t_i}^{t_f} dt \frac{1}{2}m\dot{\xi}^2 - \frac{1}{2}\int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt' \left.\frac{\partial^2 V}{\partial q(t)\partial q(t')}\right|_{q_c} \xi(t)\xi(t') - O(\xi^3) \tag{5.37}$$

Once again, the boundary conditions $\xi(t_i) = \xi(t_f) = 0$ and the fact the $q_c(t)$ is a solution of the equation of motion together imply that the last two terms of Eq. (5.36) vanish identically.

Thus, to the extent that we are allowed to neglect the $O(\xi^3)$ corrections (and higher), the effective action $S_{\text{eff}}$ can be approximated by an action that is quadratic in the fluctuation $\xi$. In general, this effective action will depend on the actual classical trajectory, since in general $V''(q_c)$ is not constant but is a function of time determined by $q_c(t)$. However, if one is interested in the quantum fluctuations about a *minimum* of the potential $V(q)$, then $q_c(t)$ is constant (and equal to the minimum). We will discuss below this case in detail.

Before we embark in an actual computation it is worthwhile to ask when it should be a good approximation to neglect the terms $O(\xi^3)$ (and higher). Since we are expanding about the classical path $q_c$, we expect that this approximation should be correct as we formally take the limit $\hbar \to 0$. In the path integral the effective action always appears in the combination $S_{\text{eff}}/\hbar$. Hence, for an effective action that is quadratic in $\xi$, we can eliminate the dependence on $\hbar$ by the rescaling

$$\xi = \sqrt{\hbar}\, \tilde{\xi} \tag{5.38}$$

This rescaling leaves the classical contribution $S(q_c)/\hbar$ unaffected. However, terms with powers higher than quadratic in $\xi$, say $O(\tilde{\xi}^n)$, scale like $\hbar^{n/2}$. Thus the action (divided by $\hbar$) has an expansion of the form

$$\frac{S}{\hbar} = \frac{1}{\hbar} S^{(0)}(q_c) + S^{(2)}(\tilde{\xi}; q_c) + \sum_{n=3}^{\infty} \hbar^{n/2} S^{(n)}(\tilde{\xi}; q_c) \tag{5.39}$$

Thus, in the limit $\hbar \to 0$, we can formally expand the weight of the path integral in powers of $\hbar$. The matrix element we are calculating then takes the form

$$\langle q_f, t_f | q_i, t_i \rangle = e^{iS^{(0)}(q_c)/\hbar}\, \mathcal{Z}^{(2)}(q_c)\, \left(1 + O(\hbar)\right) \tag{5.40}$$

The quantity $\mathcal{Z}^{(2)}(q_c)$ is the result of keeping only the quadratic approximation. The higher order terms are a power series expansion in $\hbar$ and are analytic functions of $\hbar$. Here, I have used the fact that, by symmetry, in most cases of interest the odd powers in $\xi$ in general do not contribute, although there are some cases where they do.

Let us now calculate the effect of the quantum fluctuations to quadratic order. This is equivalent to the WKB approximation. Let us denote this factor by $\mathcal{Z}$,

$$\mathcal{Z}^{(2)}(q_c) = \int_{\tilde{\xi}(t_i)=\tilde{\xi}(t_f)=0} \mathcal{D}\xi \; e^{iS^{(2)}_{\text{eff}}(\tilde{\xi},\dot{\tilde{\xi}};q_c)} \tag{5.41}$$

It is elementary to show that, due to the boundary conditions, the action $S_{\text{eff}}(\xi,\dot{\xi})$ becomes

$$S_{\text{eff}}(\tilde{\xi},\dot{\tilde{\xi}}) = \frac{1}{2}\int_{t_i}^{t_f} dt\; \tilde{\xi}(t) \left[-m\,\frac{d^2}{dt^2} - V''(q_c(t))\right]\tilde{\xi}(t) \tag{5.42}$$

The differential operator

$$\hat{A} = -m\frac{d^2}{dt^2} - V''(q_c(t)) \tag{5.43}$$

has the form of a Schrödinger operator for a particle on a "coordinate" $t$ in a potential $-V''(q_c(t))$. Let $\psi_n(t)$ be a complete set of eigenfunctions of $\hat{A}$ satisfying the boundary conditions $\psi(t_i) = \psi(t_f) = 0$. Completeness and orthonormality implies that the eigenfunctions $\{\psi_n(t)\}$ satisfy

$$\sum_n \psi_n^*(t)\psi_n(t') = \delta(t-t'), \qquad \int_{t_i}^{t_f} dt\; \psi_n^*(t)\psi_m(t) = \delta_{n,m} \tag{5.44}$$

An arbitrary function $\tilde{\xi}(t)$ that satisfies the vanishing boundary conditions of Eq.(5.33) can be expanded as a linear combination of the basis eigenfunctions $\{\psi_n(t)\}$,

$$\tilde{\xi}(t) = \sum_n c_n\psi_n(t) \tag{5.45}$$

Clearly, we have $\tilde{\xi}(t_i) = \tilde{\xi}(t_f) = 0$ as we should.

For the special case of $q_i = q_f = q_0$, where $q_0$ is a minimum of the potential $V(q)$, $V''(q_0) = \omega_{\text{eff}} > 0$ is a constant, and the eigenvectors of the Schrödinger operator are just plane waves. For a linear harmonic oscillator $\omega_{\text{eff}} = \omega$. Thus, in this case the eigenvectors are

$$\psi_n(t) = b_n \sin(k_n(t-t_i)) \tag{5.46}$$

where

$$k_n = \frac{\pi n}{t_f - t_i} \qquad n = 1,2,3,\ldots \tag{5.47}$$

and $b_n = 1/\sqrt{t_f - t_i}$. The eigenvalues of $\hat{A}$ are

$$A_n = k_n^2 - \omega_{\text{eff}}^2 = \frac{\pi^2}{(t_f-t_i)^2}n^2 - \omega_{\text{eff}}^2 \tag{5.48}$$

By using the expansion of Eq. (5.45), we find that the action $S^{(2)}$ takes the form

$$S^{(2)} = \frac{1}{2}\int_{t_i}^{t_f} dt\, \tilde{\xi}(t)\, \hat{A}\, \tilde{\xi}(t) = \frac{1}{2}\sum_n A_n c_n^2 \tag{5.49}$$

where we have used the completeness and orthonormality of the basis functions $\{\psi_n(t)\}$.

The expansion of Eq.(5.45) is a canonical transformation $\tilde{\xi}(t) \to c_n$. More to the point, the expansion is actually a *parametrization* of the possible histories in terms of a set of orthonormal functions, and it can be used to define the integration measure to be

$$\mathcal{D}\tilde{\xi} = \mathcal{N}\prod_n \frac{dc_n}{\sqrt{2\pi}} \tag{5.50}$$

with unit Jacobian. Here $\mathcal{N}$ is an irrelevant normalization constant that will be defined below.

Finally, the (formal) Gaussian integral, which is defined by a suitable analytic continuation procedure, is

$$\int_{-\infty}^{\infty} \frac{dc_n}{\sqrt{2\pi}}\, e^{\frac{i}{2}A_n c_n^2} = \left[-iA_n\right]^{-1/2} \tag{5.51}$$

can be used to write the amplitude as

$$\mathcal{Z}^{(2)} = \mathcal{N}\prod_n A_n^{-1/2} \equiv \mathcal{N}\,(\mathrm{Det}\hat{A})^{-1/2} \tag{5.52}$$

where we have used the *definition* that the determinant of an *operator* is equal to the product of its eigenvalues. Therefore, up to a normalization constant, we obtained the result

$$\mathcal{Z}^{(2)} = (\mathrm{Det}\hat{A})^{-1/2} \tag{5.53}$$

We have thus reduced the problem of the computation of the leading (Gaussian) fluctuations to the path-integral to the computation of a determinant of the fluctuation operator, a differential operator defined by the choice of classical trajectory. Below we will see how this is done.

### *5.2.1 Analytic continuation to imaginary time*

It is useful to consider the related problem obtained by an analytic continuation to *imaginary time*, $t \to -i\tau$. We saw before that there is a relation between this problem and Statistical Physics. We will now work out one example that will be very instructive.

Formally, upon the analytic continuation $t \to -i\tau$, the matrix element of the time evolution operator becomes

$$\langle q_f | e^{-\frac{i}{\hbar} H(t_f - t_i)} | q_i \rangle \to \langle q_f | e^{-\frac{1}{\hbar} H(\tau_f - \tau_i)} | q_i \rangle \tag{5.54}$$

Let us choose

$$\tau_i = 0 \qquad \tau_f = \beta\hbar \tag{5.55}$$

where $\beta = 1/T$, and $T$ is the temperature (in units of $k_B = 1$). Hence, we find that

$$\langle q_f, -i\beta/\hbar | q_i, 0 \rangle = \langle q_f | e^{-\beta H} | q_i \rangle \tag{5.56}$$

The operator $\hat{\rho}$

$$\hat{\rho} = e^{-\beta H} \tag{5.57}$$

is the *Density Matrix* in the Canonical Ensemble of Statistical Mechanics for a system with Hamiltonian $H$ in thermal equilibrium at temperature $T$.

It is customary to define the *Partition Function* $\mathcal{Z}$,

$$\mathcal{Z} = \operatorname{tr} e^{-\beta H} \equiv \int dq \, \langle q | e^{-\beta H} | q \rangle \tag{5.58}$$

where I inserted a complete set of eigenstates of $\hat{q}$. Using the results that were derived above, we see that the partition function $\mathcal{Z}$ can be written as a (Euclidean) *Feynman path integral in imaginary time*, of the form

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}q[\tau] \exp\left\{ -\frac{1}{\hbar} \int_0^{\beta\hbar} d\tau \left[ \frac{1}{2} m \left( \frac{\partial q}{\partial \tau} \right)^2 + V(q) \right] \right\} \\ &\equiv \int \mathcal{D}q[\tau] \exp\left\{ - \int_0^{\beta} d\tau \left[ \frac{m}{2\hbar^2} \left( \frac{\partial q}{\partial \tau} \right)^2 + V(q) \right] \right\} \end{aligned} \tag{5.59}$$

where, in the last equality we have rescaled $\tau \to \tau/\hbar$.

Since the Partition Function is a *trace* over states, we must use boundary conditions such that the initial and final states are the same state, and to sum over all such states. In other words, we must have *periodic* boundary conditions in imaginary time (PBC's),

$$q(\tau) = q(\tau + \beta) \tag{5.60}$$

Therefore, a quantum mechanical system at finite temperature $T$ can be described in terms of an equivalent system in classical statistical mechanics with Hamiltonian (or energy)

$$\mathcal{H} = \frac{m}{2\hbar^2}\left(\frac{\partial q}{\partial \tau}\right)^2 + V(q) \tag{5.61}$$

on a segment of length $1/T$ and obeying PBC's. This effectively means that the segment is actually a ring of length $\beta = 1/T$.

Alternatively, upon inserting a complete set of eigenstates of the Hamiltonian, it is easy to see that an arbitrary matrix element of the density matrix has the form

$$\begin{aligned}\langle q'|e^{-\beta H}|q\rangle &= \sum_{n=0}^{\infty}\langle q'|n\rangle\langle n|q\rangle e^{-\beta E_n}\\ &= \sum_{n=0}^{\infty} e^{-\beta E_n}\psi_n^*(q')\psi_n(q) \xrightarrow[\beta\to\infty]{} e^{-\beta E_0}\psi_0^*(q')\psi_0(q)\end{aligned} \tag{5.62}$$

where $\{E_n\}$ are the eigenvalues of the Hamiltonian, $E_0$ is the ground state energy and $\psi_0(q)$ is the ground state wave function.

Therefore, we can calculate both the ground state energy $E_0$ and the ground state wave function from the density matrix and consequently from the (imaginary time) path integral. For example, from the identity

$$E_0 = -\lim_{\beta\to\infty}\frac{1}{\beta}\ln \operatorname{tr} e^{-\beta H} \tag{5.63}$$

we see that the ground state energy is given by

$$E_0 = -\lim_{\beta\to\infty}\frac{1}{\beta}\ln\int_{q(0)=q(\beta)}\mathcal{D}q\ \exp\left\{-\int_0^\beta d\tau\left[\frac{m}{2\hbar^2}\left(\frac{\partial q}{\partial\tau}\right)^2 + V(q)\right]\right\} \tag{5.64}$$

Mathematically, the imaginary time path integral is a better behaved object than its real time counterpart, since it is a sum of positive quantities, the statistical weights. In contrast, the Feynman path integral (in real time) is a sum of phases and, as such, is an ill-defined object. It is actually conditionally convergent, and to make sense of it convergence factors (or regulators) will have to be introduced. The effect of these convergence factors is actually an analytic continuation to imaginary time. We will encounter the same problem in the calculation of propagators. Thus, the imaginary time path integral, often referred to as the Euclidean path integral (as opposed to Minkowski), can be used to describe both a quantum system and a statistical mechanics system.

Finally, we notice that at low temperatures $T \to 0$, the Euclidean path integral can be approximated using methods similar to the ones we discussed for the (real time) Feynman Path Integral. The main difference is that we must sum over trajectories which are periodic in imaginary time with period $\beta = 1/T$. In practice this sum can only be done exactly for simple systems such as the harmonic oscillator, and for more general systems one has to resort to some form of perturbation theory. Here we will consider a physical system described by a dynamical variable $q$ and a potential energy $V(q)$ which has a minimum at $q_0 = 0$. For simplicity we will take $V(0) = 0$ and we will denote by $m\omega^2 = V''(0)$ (in other words, an effective harmonic oscillator). The partition function is given by the Euclidean path integral

$$\mathcal{Z} = \int \mathcal{D}q[\tau] \exp\left(-\frac{1}{2}\int_0^\beta \xi(\tau)\hat{A}_E\xi(\tau)d\tau\right) \tag{5.65}$$

where $\hat{A}_E$ is the imaginary time, or Euclidean, version of the operator $\hat{A}$, and it is given by

$$\hat{A}_E = -\frac{m}{\hbar^2}\frac{d^2}{d\tau^2} + V''(q_c(\tau)) \tag{5.66}$$

The functions this operator acts on obey periodic boundary conditions with period $\beta$. Notice the important change in the sign of the term of the potential. Hence, once again we will need to compute a functional determinant, although the operator now acts on functions obeying periodic boundary conditions. In a later chapter we will see that in the case of *fermionic* theories, the boundary conditions become *antiperiodic*.

### 5.2.2 The functional determinant

We will now do the computation of the determinant in $\mathcal{Z}^{(2)}$. We will do the calculation in imaginary time and then we will carry out the analytic continuation to real time. We will follow closely the method is explained in detail in Sidney Coleman's book, (Coleman, 1985).

We want to compute

$$D = \mathrm{Det}\left[-\frac{m}{\hbar^2}\frac{d^2}{d\tau^2} + V''(q_c(\tau))\right] \tag{5.67}$$

subject to the requirement that the space of functions that the operator acts on obeys specific boundary conditions in (imaginary) time. We will be interested in two cases: (a) Vanishing Boundary Conditions (VBC's), which are useful to study quantum mechanics at $T = 0$, and (b) Periodic Boundary Conditions (PBC's) with period $\beta = 1/T$. The approach is somewhat different in the two situations.

*A: vanishing boundary conditions*

We define the (real) variable $x = \frac{\hbar}{m}\tau$. The range of $x$ is the interval $[0, L]$, with $L = \hbar\beta/\sqrt{m}$. Let us consider the following eigenvalue problem for the Schrödinger operator $-\partial^2 + W(x)$,

$$\left(-\partial^2 + W(x)\right)\psi(x) = \lambda\psi(x) \tag{5.68}$$

subject to the boundary conditions $\psi(0) = \psi(L) = 0$. Formally, the determinant is given by

$$D = \prod_n \lambda_n \tag{5.69}$$

where $\{\lambda_n\}$ is the spectrum of eigenvalues of the operator $-\partial^2 + W(x)$ for a space of functions satisfying a given boundary condition.

Let us define an auxiliary function $\psi_\lambda(x)$, with $\lambda$ a real number not necessarily in the spectrum of the operator, such that the following requirements are met:

a. $\psi_\lambda(x)$ is a solution of Eq. (5.68), and
b. $\psi_\lambda$ obeys the *initial* conditions, $\psi_\lambda(0) = 0$ and $\partial_x\psi_\lambda(0) = 1$.

It is easy to see that $-\partial^2 + W(x)$ has an eigenvalue at $\lambda_n$ if and only if $\psi_{\lambda_n}(L) = 0$. (Because of this property this procedure is known as the *shooting method*.) Hence, the determinant $D$ of Eq. (5.69) is equal to the product of the *zeros* of $\psi_\lambda(x)$ at $x = L$.

Consider now two potentials $W^{(1)}$ and $W^{(2)}$, and the associated functions, $\psi_\lambda^{(1)}(x)$ and $\psi_\lambda^{(2)}(x)$. Let us show that

$$\frac{\text{Det}\left[-\partial^2 + W^{(1)}(x) - \lambda\right]}{\text{Det}\left[-\partial^2 + W^{(2)}(x) - \lambda\right]} = \frac{\psi_\lambda^{(1)}(L)}{\psi_\lambda^{(2)}(L)} \tag{5.70}$$

The l. h. s. of Eq. (5.70) is a meromorphic function of $\lambda$ in the complex plane, which has simple zeros at the eigenvalues of $-\partial^2+W^{(1)}(x)$ and simple poles at the eigenvalues of $-\partial^2 + W^{(2)}(x)$. Also, the l. h. s. of Eq. (5.70) approaches 1 as $|\lambda| \to \infty$, except along the positive real axis which is where the spectrum of eigenvalues of both operators is. Here we have assumed that the eigenvalues of the operators are non-degenerate, which is the general case. Similarly, the right hand side of Eq. (5.70) is also a meromorphic function of $\lambda$, which has *exactly the same zeros and the same poles* as the left hand side. It also goes to 1 as $|\lambda| \to \infty$ (again, except along the positive real axis), since the wave-functions $\psi_\lambda$ are asymptotically plane waves in this limit . Therefore, the function formed by taking the ratio r. h. s. / l. h.

s. is an *analytic* function on the *entire complex plane* and it approaches 1 as $|\lambda| \to \infty$. Then, general theorems of the theory of functions of a complex variable tell us that this function is *equal* to 1 everywhere.

From these considerations we conclude that the following ratio is independent of $W(x)$,

$$\frac{\text{Det}\left(-\partial^2 + W(x) - \lambda\right)}{\psi_\lambda(L)} \tag{5.71}$$

We now define a constant $\mathcal{N}$ such that

$$\frac{\text{Det}\left(-\partial^2 + W(x)\right)}{\psi_0(L)} = \pi\hbar\mathcal{N}^2 \tag{5.72}$$

Then, we can write

$$\mathcal{N}\left[\text{Det}\left(-\partial^2 + W\right)\right]^{-1/2} = \left[\pi\hbar\psi_0(L)\right]^{-1/2} \tag{5.73}$$

Thus we reduced the computation of the determinant, including the normalization constant, to finding the function $\psi_0(L)$. For the case of the linear harmonic oscillator, this function is the solution of

$$\left[-\frac{\partial^2}{\partial x^2} + m\omega^2\right]\psi_0(x) = 0 \tag{5.74}$$

with the initial conditions, $\psi_0(0) = 0$ and $\psi_0'(0) = 1$. The solution is

$$\psi_0(x) = \frac{1}{\sqrt{m}\omega}\sinh(\sqrt{m}\omega x) \tag{5.75}$$

Hence,

$$\mathcal{Z} = \mathcal{N}\left[\text{Det}\left(-\frac{\partial^2}{\partial x^2} + m\omega^2\right)\right]^{-1/2} = \left[\pi\hbar\psi_0(L)\right]^{-1/2} \tag{5.76}$$

and we find

$$\mathcal{Z} = \left[\frac{\pi\hbar}{\sqrt{m}\omega}\sinh(\beta\omega)\right]^{-1/2} \tag{5.77}$$

where we have used $L == \hbar\beta/\sqrt{m}$. From this result we find that the ground state energy is

$$E_0 = \lim_{\beta\to\infty}\frac{-1}{\beta}\ln\mathcal{Z} = \frac{\hbar\omega}{2} \tag{5.78}$$

as it should be.

Finally, by means of an analytic continuation back to real time, we can use these results to find, for instance, the amplitude to return to the origin after some time $T$. Thus, for $t_f - t_i = T$ and $q_f = q_i = 0$, we get

$$\langle 0, T | 0, 0 \rangle = \left[ \frac{i\pi\hbar}{\sqrt{m\omega}} \sin(\omega T) \right]^{-1/2} \tag{5.79}$$

*B: periodic boundary conditions*

Periodic boundary conditions imply that the histories satisfy $q(\tau) = q(\tau+\beta)$. Hence, these functions can be expanded in a Fourier series of the form

$$q(\tau) = \sum_{n=-\infty}^{\infty} e^{i\omega_n \tau} q_n \tag{5.80}$$

where $\omega_n = 2\pi n/\beta$. Since $q(\tau)$ is real, we have the constraint $q_{-n} = q_n^*$. For such configurations (or histories) the action becomes

$$\begin{aligned} S = & \int_0^\beta d\tau \left[ \frac{m}{2\hbar^2} \left( \frac{\partial q}{\partial \tau} \right)^2 + \frac{1}{2} V''(0) q^2 \right] \\ = & \frac{\beta}{2} V''(0) q_0^2 + \beta \sum_{n \geq 1} \left[ \frac{m}{\hbar^2} \omega_n^2 + V''(0) \right] |q_n|^2 \end{aligned} \tag{5.81}$$

The integration measure now is

$$\mathcal{D}q[\tau] = \mathcal{N} \frac{dq_0}{\sqrt{2\pi}} \prod_{n \geq 1} \frac{d\mathrm{Re}q_n \, d\mathrm{Im}q_n}{2\pi} \tag{5.82}$$

where $\mathcal{N}$ is a normalization constant that will be discussed below. After doing the Gaussian integrals, the partition function becomes,

$$\mathcal{Z} = \mathcal{N} \frac{1}{\sqrt{\beta V''(0)}} \prod_{n \geq 1} \frac{1}{\frac{\beta m}{\hbar^2} \omega_n^2 + \beta V''(0)} = \mathcal{N} \left[ \prod_{n=-\infty}^{\infty} \frac{1}{\frac{\beta m}{\hbar^2} \omega_n^2 + \beta V''(0)} \right]^{1/2} \tag{5.83}$$

Formally, the infinite product that enters in this equation is divergent. The normalization constant $\mathcal{N}$ eliminates this divergence. This is an example of what is called a *regularization*. The regularized partition function is

$$\mathcal{Z} = \sqrt{\frac{m}{\hbar^2 \beta}} \frac{1}{\sqrt{\beta V''(0)}} \prod_{n \geq 1} \left[ 1 + \frac{\hbar^2 V''(0)}{m \omega_n^2} \right]^{-1} \tag{5.84}$$

Using the identity

$$\prod_{n \geq 1} \left( 1 + \frac{a^2}{n^2 \pi^2} \right) = \frac{\sinh a}{a} \tag{5.85}$$

we find

$$\mathcal{Z} = \frac{1}{2\sinh\left(\frac{\beta\hbar}{2}\left(\frac{V''(0)}{m}\right)^{1/2}\right)} \tag{5.86}$$

which is the partition function for a linear harmonic oscillator, see L. D. Landau and E. M. Lifshitz, *Statistical Physics*, (Landau and Lifshitz, 1959b).

## 5.3 Path integrals for a scalar field theory

We will now develop the path-integral quantization picture for a scalar field theory. Our starting point will be the canonically quantized scalar field. As we saw before, in canonical quantization the scalar field $\hat{\phi}(x)$ is an operator that acts on a Hilbert space of states. We will use the field representation, which is the analog of the conventional coordinate representation in Quantum Mechanics.

Thus, the basis states are labelled by the field configuration at some fixed time $x_0$, a set of states of the form $\{ \ |\{\phi(\boldsymbol{x}, x_0)\}\rangle \ \}$. The field operator $\hat{\phi}(\boldsymbol{x}, x_0)$ acts trivially on these states,

$$\hat{\phi}(\boldsymbol{x}, x_0)|\{\phi(\boldsymbol{x}, x_0)\}\rangle = \phi(\boldsymbol{x}, x_0)|\{\phi(\boldsymbol{x}, x_0)\}\rangle \tag{5.87}$$

The set of states $\{ \ |\{\phi(\boldsymbol{x}, x_0)\}\rangle \ \}$ is both complete and orthonormal. Completeness here means that these states span the entire Hilbert space. Consequently the identity operator $\hat{\mathcal{I}}$ in the full Hilbert space can be expanded in a complete basis in the usual manner, which for this basis it means

$$\hat{\mathcal{I}} = \int \mathcal{D}\phi(\boldsymbol{x}, x_0) \ |\{\phi(\boldsymbol{x}, x_0)\}\rangle\langle\{\phi(\boldsymbol{x}, x_0)\}| \tag{5.88}$$

Since the completeness condition is a sum over all the states in the basis and since this basis is the set of field configurations at a given time $x_0$, we will need to give a definition for integration measure which represents the sums over the field configurations. In this case, the definition of the integration measure is trivial,

$$\mathcal{D}\phi(\boldsymbol{x}, x_0) = \prod_{\boldsymbol{x}} d\phi(\boldsymbol{x}, x_0) \tag{5.89}$$

Likewise, orthonormality of the basis states is the condition

$$\langle\phi(\boldsymbol{x}, x_0)|\phi'(\boldsymbol{x}, x_0)\rangle = \prod_{\boldsymbol{x}} \delta\left(\phi(\boldsymbol{x}, x_0) - \phi'(\boldsymbol{x}, x_0)\right) \tag{5.90}$$

Thus, we have a working definition of the Hilbert space for a real scalar field.

In canonical quantization, the classical canonical momentum $\Pi(\boldsymbol{x}, x_0)$, defined as

$$\Pi(\boldsymbol{x}, x_0) = \frac{\delta \mathcal{L}}{\delta \partial_0 \phi(\boldsymbol{x}, x_0)} = \partial_0 \phi(\boldsymbol{x}, x_0) \tag{5.91}$$

becomes an operator that acts on the same Hilbert space as the field itself $\phi$ does. The field operator $\hat{\phi}(x)$ and the canonical momentum operator $\hat{\Pi}(x)$ satisfy *equal-time canonical commutation relations*

$$\left[\hat{\phi}(\boldsymbol{x}, x_0), \hat{\Pi}(\boldsymbol{y}, x_0)\right] = i\hbar \delta^3(\boldsymbol{x} - \boldsymbol{y}) \tag{5.92}$$

Here we will consider a real scalar field whose Lagrangian density is

$$\mathcal{L} = \frac{1}{2} \left(\partial_\mu \phi\right)^2 - V(\phi) \tag{5.93}$$

It is a simple matter to generalize what follows below to more general cases, such as complex fields and/or several components. Let us also recall that the Hamiltonian for a scalar field is given by

$$\hat{H} = \int d^3x \left[\frac{1}{2}\hat{\Pi}^2(x) + \frac{1}{2}\left(\nabla \hat{\phi}(x)\right)^2 + V(\hat{\phi}(x))\right] \tag{5.94}$$

For reasons that will become clear soon, it is convenient to add an extra term to the Lagrangian density of the scalar field, Eq. (5.93), of the form

$$\mathcal{L}_{\text{source}} = J(x)\, \phi(x) \tag{5.95}$$

The field $J(x)$ is called an *external source*. The field $J(x)$ is the analog of external *forces* acting on a system of classical particles. Here we will always assume that the sources $J(x)$ vanish both at spacial infinity (at all times) and everywhere in both the remote past and in the remote future,

$$\lim_{|\boldsymbol{x}| \to \infty} J(\boldsymbol{x}, x_0) = 0 \qquad \lim_{x_0 \to \pm\infty} J(\boldsymbol{x}, x_0) = 0 \tag{5.96}$$

The total Lagrangian density is

$$\mathcal{L}(\phi, J) = \mathcal{L} + \mathcal{L}_{\text{source}} \tag{5.97}$$

Since the source $J(x)$ is in general a function of space and time, the Hamiltonian that follows from this Lagrangian is formally time-dependent.

We will derive the path integral for this quantum field theory by following the same procedure we used for the case of a finite quantum mechanical system. Hence we begin by considering the Wightman function defined as the amplitude

$$_J\langle \{\phi(\boldsymbol{x}, x_0)\} | \{\phi'(\boldsymbol{y}, y_0)\} \rangle_J \tag{5.98}$$

In other words, we want the transition amplitude in the background of the sources $J(x)$. We will be interested in situations in which $x_0$ is in the remote future and $y_0$ is in the remote past. It turns out that this amplitude is intimately related to the computation of ground state (or *vacuum*) expectation values of *time ordered* products of field operators in the Heisenberg representation

$$G^{(N)}(x_1, \ldots, x_N) \equiv \langle 0|T[\hat{\phi}(x_1)\ldots\hat{\phi}(x_N)]|0\rangle \tag{5.99}$$

which are known as the $N$-point functions (or correlators). In particular the 2-point function

$$G^{(2)}(x_1 - x_2) \equiv -i\langle 0|T[\hat{\phi}(x_1)\hat{\phi}(x_2)]|0\rangle \tag{5.100}$$

is called the *Feynman Propagator* for this theory. We will see later on that all quantities of physical interest can be obtained from a suitable correlation function of the type of Eq. (5.99).

In Eq. (5.99) we have use the notation $T[\hat{\phi}(x_1)\ldots\hat{\phi}(x_N)]$ for the *time-ordered product* of Heisenberg field operators. For *any* pair Heisenberg of operators $\hat{A}(x)$ and $\hat{B}(y)$, that commute for space-like separations, their time-ordered product is defined to be

$$T[\hat{A}(x)\hat{B}(y)] = \theta(x_0 - y_0)\hat{A}(x)\hat{B}(y) + \theta(y_0 - x_0)\hat{B}(y)\hat{A}(x) \tag{5.101}$$

where $\theta(x)$ is the step (or Heaviside) function

$$\theta(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{otherwise} \end{cases} \tag{5.102}$$

This definition is generalized by induction to to the product of any number of operators. Notice that inside a time-ordered product the Heisenberg operators behave as if they were c-numbers.

Let us now recall the structure of the derivation that we gave of the path integral in Quantum Mechanics. We will paraphrase that derivation for this field theory. We considered an amplitude equivalent to Eq. (5.98), and realized that this amplitude is actually a matrix element of the evolution operator,

$$_J\langle\{\phi(\boldsymbol{x}, x_0)\}|\{\phi'(\boldsymbol{y}, y_0)\}\rangle_J = \langle\{\phi(\boldsymbol{x})\}|T\, e^{-\frac{i}{\hbar}\int_{y_0}^{x_0} dx_0'\, \widehat{H}(x_0')}|\{\phi'(\boldsymbol{y})\}\rangle \tag{5.103}$$

where $T$ stands for the time ordering symbol (not temperature!), and $\widehat{H}(x_0')$

is the time-dependent Hamiltonian whose Hamiltonian density is

$$\widehat{\mathcal{H}}(x_0) = \frac{1}{2}\widehat{\Pi}^2(\boldsymbol{x}, x_0) + \frac{1}{2}\left(\nabla\hat{\phi}(\boldsymbol{x}, x_0)\right)^2 + V(\hat{\phi}(\boldsymbol{x}, x_0)) - J(\boldsymbol{x}, x_0)\hat{\phi}(\boldsymbol{x}, x_0) \tag{5.104}$$

Paraphrasing the construction used in the case of Quantum Mechanics of a particle, we first partition the time interval in a large number of steps $N$, each of width $\Delta t$, and then insert a complete set of eigenstates of the field operator $\hat{\phi}$, since the field plays the role of the coordinate. As it turned out, we also had to insert complete sets of eigenstates of the canonical momentum operator, which here means the canonical field operator $\widehat{\Pi}(\boldsymbol{x})$. Upon formally taking the time-continuum limit, $N \to \infty$ and $\Delta t \to 0$ while keeping $N\Delta t$ fixed, we obtain the result that the *phase-space* path integral of the field theory is

$$_J\langle\{\phi(\boldsymbol{x}, x_0)\}|\{\phi'(\boldsymbol{y}, y_0)\}\rangle_J = \int_{\text{b. c.}} \mathcal{D}\phi\mathcal{D}\Pi\; e^{\frac{i}{\hbar}\int d^4x\left[\dot{\phi}\Pi - \mathcal{H}(\phi, \Pi) + J\phi\right]} \tag{5.105}$$

where b.c. indicates the boundary conditions specified by the requirement that the initial and final states be $|\{\phi(\boldsymbol{x}, x_0)\}\rangle$ and $|\{\phi'(\boldsymbol{y}, y_0)\}\rangle$, respectively.

Exactly as in the case of the path integral for a particle, the Hamiltonian of this theory is quadratic in the canonical momenta $\Pi(x)$. Hence, we can further integrate out the field $\Pi(x)$, and obtain the Feynman path integral for the scalar field theory in the form of a *sum over histories* of field configurations:

$$_J\langle\{\phi(\boldsymbol{x}, x_0)\}|\{\phi'(\boldsymbol{y}, y_0)\}\rangle_J = \mathcal{N}\int_{\text{b. c.}} \mathcal{D}\phi\; e^{\frac{i}{\hbar}S(\phi, \partial_\mu\phi, J)} \tag{5.106}$$

where $\mathcal{N}$ is an (unimportant) normalization constant, and $S(\phi, \partial_\mu\phi, J)$ is the action for a real scalar field $\phi(x)$ coupled to a source $J(x)$,

$$S(\phi, \partial_\mu\phi, J) = \int d^4x\left[\frac{1}{2}\left(\partial_\mu\phi\right)^2 - V(\phi) + J\phi\right] \tag{5.107}$$

## 5.4 Path integrals and propagators

In Quantum Field Theory we will be interested in calculating vacuum (ground state) expectation values of field operators at various space-time locations. Thus, instead of the amplitude $_J\langle\{\phi(\boldsymbol{x}, x_0)\}|\{\phi'(\boldsymbol{y}, y_0)\}\rangle_J$ we may be interested in a transition between an initial state, at $y_0 \to -\infty$ which is the *vacuum state* $|0\rangle$, i.e. the ground state of the scalar field in the absence of the source $J(x)$, and a final state at $x_0 \to \infty$ which is also the vacuum state of the theory in the absence of sources. We will denote this matrix element by

$$Z[J] = {}_J\langle 0|0\rangle_J \tag{5.108}$$

This matrix element is called the *vacuum persistence amplitude*.

Let us see now how the vacuum persistence amplitude is related to the Feynman path integral for a scalar field of Eq. (5.106). In order to do that we will assume that the source $J(x)$ is "on" between times $t < t'$ and that we watch the system on a much longer time interval $T < t < t' < T'$. For this interval, we can now use the Superposition Principle to insert complete sets of states at intermediate times $t$ and $t'$, and write the amplitude in the form

$$\begin{aligned} &{}_J\langle\{\Phi'(\boldsymbol{x},T')\}|\{\Phi(\boldsymbol{x},T)\}\rangle_J = \\ &\quad\int \mathcal{D}\phi(\boldsymbol{x},t)\,\mathcal{D}\phi'(\boldsymbol{x},t')\langle\{\Phi'(\boldsymbol{x},T')\}|\{\phi'(\boldsymbol{x},t')\}\rangle \\ &\qquad\qquad \times {}_J\langle\{\phi'(\boldsymbol{x},t')\}|\{\phi(\boldsymbol{x},t)\}\rangle_J\langle\{\phi(\boldsymbol{x},t)\}|\{\Phi(\boldsymbol{x},T)\}\rangle \end{aligned} \tag{5.109}$$

The matrix elements $\langle\{\Phi'(\boldsymbol{x},T')\}|\{\phi'(\boldsymbol{x},t')\}\rangle$ and $\langle\{\phi(\boldsymbol{x},t)\}|\{\Phi(\boldsymbol{x},T)\}\rangle$ are given by

$$\begin{aligned} \langle\{\phi(\boldsymbol{x},t)\}|\{\Phi(\boldsymbol{x},T)\}\rangle &= \sum_n \Psi_n[\{\phi(\boldsymbol{x})\}]\Psi_n^*[\{\Phi(\boldsymbol{x})\}]\; e^{-iE_n(t-T)/\hbar} \\ \langle\{\Phi'(\boldsymbol{x},T')\}|\{\phi'(\boldsymbol{x},t')\}\rangle &= \sum_m \Psi_m[\{\Phi'(\boldsymbol{x})\}]\Psi_m^*[\{\phi'(\boldsymbol{x})\}]\; e^{-iE_m(T'-t')/\hbar} \end{aligned} \tag{5.110}$$

where we have introduced complete sets of eigenstates $|\{\Psi_n\}\rangle$ of the Hamiltonian of the scalar field (without sources) and the corresponding wave functions, $\{\Psi_n[\Phi(\boldsymbol{x})]\}$.

At long times $T$ and $T'$ these series expansions oscillate very rapidly and a definition must be provided to make sense on these expressions. To this end, we will now analytically continue $T$ along the *positive* imaginary time axis, and $T'$ along the *negative* imaginary time axis, as shown in figure 5.4. After carrying out the analytic continuation, we find that the following identities

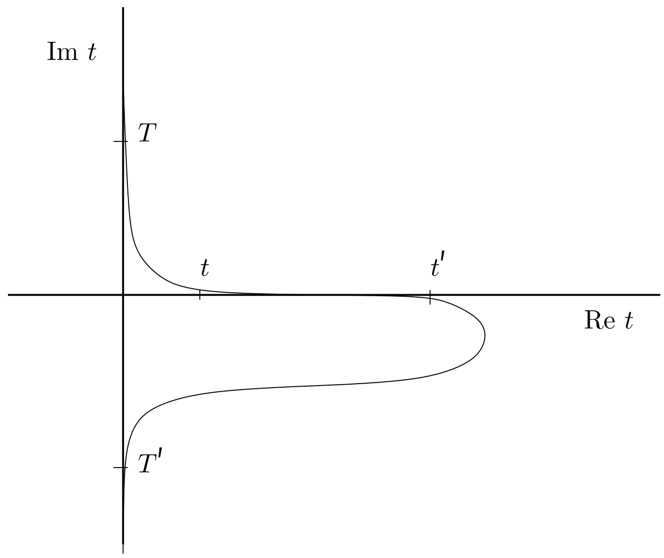


Figure 5.4 Analytic continuation.

hold,

$$\lim_{T \to +i\infty} e^{-iE_0 T/\hbar} \langle \{\phi(\boldsymbol{x}, t)\} | \{\Phi(\boldsymbol{x}, T)\} \rangle = \Psi_0[\{\phi\}] \, \Psi_0^*[\{\Phi\}] \, e^{-iE_0 t/\hbar}$$
$$\lim_{T' \to -i\infty} e^{iE_0 T'/\hbar} \langle \{\Phi'(\boldsymbol{x}, T')\} | \{\phi(\boldsymbol{x}, t')\} \rangle = \Psi_0[\{\Phi'\}] \, \Psi_0^*[\{\phi'\}] \, e^{iE_0 t'/\hbar} \tag{5.111}$$

This result is known as the Gell-Mann-Low Theorem.(Gell-Mann and Low, 1951) In this limit, the contributions from excited states drop out provided the vacuum state $|0\rangle$ is non-degenerate. This procedure is equivalent to the standard adiabatic turning on and off of the external sources. The restriction to a non-degenerate vacuum state can be done by lifting a possible degeneracy by means of an infinitesimally weak external perturbation, which is switched off after the infinite time limit is taken. We will encounter similar issues in our discussion of spontaneous symmetry breaking in later chapters.

Hence, in the same limit, we also find that the following relation holds

$$
\begin{aligned}
&\lim_{T\to+i\infty}\lim_{T'\to-i\infty}\frac{\langle\{\Phi'(\boldsymbol{x},T')\}|\Phi(\boldsymbol{x},T)\}\rangle}{\exp\left[-iE_0(T'-T)/\hbar\right]\ \Psi_0^*[\{\Phi\}]\ \Psi_0[\{\Phi'\}]}\\
&\qquad=\int\mathcal{D}\Phi\mathcal{D}\Phi'\ \Psi_0^*[\{\phi'(\boldsymbol{x},t')\}]\ \Psi_0[\{\phi(\boldsymbol{x},t)\}]\ {}_J\langle\{\phi'(\boldsymbol{x},t')\}|\{\phi(\boldsymbol{x},t)\}\rangle_J\\
&\qquad\equiv{}_J\langle 0|0\rangle_J
\end{aligned}
\tag{5.112}
$$

Eq.(5.112) gives us a direct relation between the Feynman Path Integral and the vacuum persistence amplitude of the form

$$
Z[J]={}_J\langle 0|0\rangle_J=\mathcal{N}\lim_{T\to+i\infty}\lim_{T'\to-i\infty}\int\mathcal{D}\phi\ e^{\frac{i}{\hbar}\int_T^{T'}d^4x\ \left[\mathcal{L}(\phi,\partial_\mu\phi)+J\phi\right]}
\tag{5.113}
$$

In other words, in this asymptotically long-time limit, the amplitude of Eq. (5.98) becomes identical to the vacuum persistence amplitude ${}_J\langle 0|0\rangle_J$, regardless of the choice of the initial and final states.

Hence we find a direct relation between the vacuum persistence function $Z[J]$ and the Feynman path integral, given by Eq. (5.113). Notice that, in this limit, we can ignore the “hard” boundary condition and work instead with free boundary conditions. Or equivalently, physical properties become independent of the initial and final conditions placed.

For these reasons, from now on we will work with the simpler expression

$$
Z[J]={}_J\langle 0|0\rangle_J=\mathcal{N}\int\mathcal{D}\phi\ e^{\frac{i}{\hbar}\int d^4x\ \left[\mathcal{L}(\phi,\partial_\mu\phi)+J\phi\right]}
\tag{5.114}
$$

This is a very useful relation. We will see now that $Z[J]$ is the generating function(al) of all the vacuum expectation values of time ordered products of fields, i.e. the correlators of the theory.

In particular, let us compute the expression

$$
\frac{1}{Z[0]}\frac{\delta^2Z[J]}{\delta J(x)\delta J(x')}\bigg|_{J=0}=\frac{1}{\langle 0|0\rangle}\frac{\delta^2\,{}_J\langle 0|0\rangle_J}{\delta J(x)\delta J(x')}\bigg|_{J=0}=\left(\frac{i}{\hbar}\right)^2\ \langle 0|T[\phi(x)\phi(x')]|0\rangle
\tag{5.115}
$$

Thus, the 2-point function, i.e. the *Feynman propagator* or *propagator* of the scalar field $\phi(x)$, becomes

$$
\langle 0|T[\phi(x)\phi(x')]|0\rangle=-i\frac{1}{\langle 0|0\rangle}\int\mathcal{D}\phi\ \phi(x)\ \phi(x')\ \exp\left(\frac{i}{\hbar}S[\phi,\partial_\mu\phi]\right)
\tag{5.116}
$$

Similarly, the $N$-point function $\langle 0|T[\phi(x_1)\dots\phi(x_N)]|0\rangle$ becomes

$$\langle 0|T[\phi(x_1)\dots\phi(x_N)]|0\rangle = (-i\hbar)^N \frac{1}{\langle 0|0\rangle} \frac{\delta^N {}_J\langle 0|0\rangle_J}{\delta J(x_1)\dots\delta J(x_N)}\Bigg|_{J=0}$$
$$= \frac{1}{\langle 0|0\rangle}\int \mathcal{D}\phi\ \phi(x_1)\dots\phi(x_N)\ \exp\left(\frac{i}{\hbar}S[\phi,\partial_\mu\phi]\right) \tag{5.117}$$

where

$$Z[0] = \langle 0|0\rangle = \int \mathcal{D}\phi\ \exp\left(\frac{i}{\hbar}S[\phi,\partial_\mu\phi]\right) \tag{5.118}$$

Therefore, we find that the path integral always yields vacuum expectation values of time-ordered products of operators. The quantity $Z[J]$ can thus be viewed as the generating functional of the correlation functions of this theory. These are actually general results that hold for the path integrals of all theories.

## 5.5 Path integrals in Euclidean space-time and Statistical Physics

In the last section we saw how to relate the computation of transition amplitudes to path integrals in Minkowski space-time with specific boundary conditions dictated by the nature of the initial and final states. In particular we derived explicit expressions for the case of fixed boundary conditions.

However we could have chosen other boundary conditions. For instance, for the amplitude to begin in any state at the initial time and to go back to the *same* state at the final time, but summing over all states. This is the same as to ask for the *trace*

$$Z'[J] = \int \mathcal{D}\Phi_J\langle\{\Phi(\boldsymbol{x},t')\}|\{\Phi(\boldsymbol{x},t)\}\rangle_J$$
$$\equiv \mathrm{Tr}\ T\ e^{-\frac{i}{\hbar}\int d^4x\,(\mathcal{H}-J\phi)}$$
$$\equiv \int_{\mathrm{PBC}} \mathcal{D}\phi\ e^{\frac{i}{\hbar}\int d^4x\,(\mathcal{L}+J\phi)} \tag{5.119}$$

where PBC stands for periodic boundary conditions on some generally *finite* time interval $t'-t$, and $T$ is the time-ordering symbol.

Let us now carry the analytic continuation to imaginary time $t\to -i\tau$, i.e. a Wick rotation. Upon a Wick rotation the theory has Euclidean invariance, i.e. rotations and translations in $D = d + 1$-dimensional space. Imaginary time plays the same role as the other $d$ spacial dimensions. Hereafter we will denote imaginary time by $x_D$, and all vectors will have indices $\mu$ that run from 1 to $D$.

We will consider two cases: infinite imaginary time interval, and finite imaginary time interval.

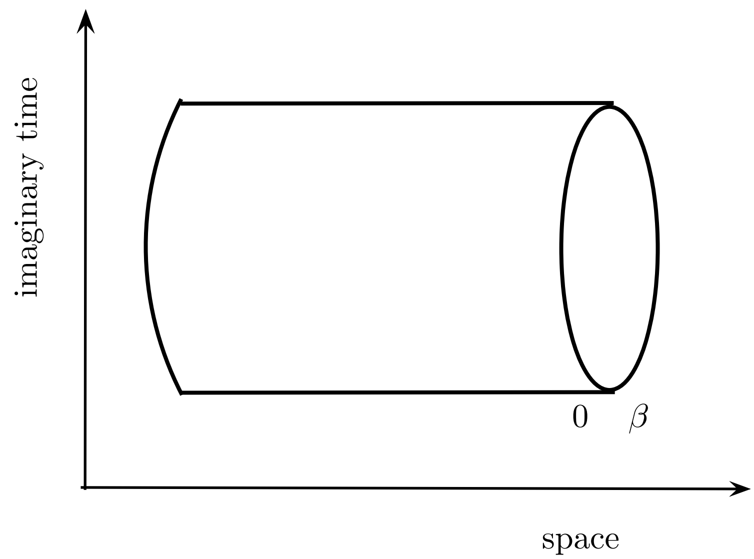


Figure 5.5 Periodic boundary conditions wraps space-time into a cylinder.

### *5.5.1 Infinite imaginary time interval*

In this case the path integral becomes

$$Z'[J] = \int \mathcal{D}\phi \; e^{-\int d^Dx \, (\mathcal{L}_E - J\phi)} \tag{5.120}$$

where $D$ is the total number of space-time dimensions. For the sake of definiteness here we discuss the four-dimensional case but the results are obviously valid more generally. Here $\mathcal{L}_E$ is the Euclidean Lagrangian

$$\mathcal{L}_E = \frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi) \tag{5.121}$$

The path integral of Eq. (5.120) has two interpretations.

One is simply the infinite time limit (in imaginary time) and therefore it must be identical to the vacuum persistence amplitude ${}_J\langle 0|0\rangle_J$. The only difference is that from here we get all the $N$-point functions in Euclidean space-time (imaginary time). Therefore, the relativistic interval is

$$x_0^2 - \boldsymbol{x}^2 \to -\tau^2 - \boldsymbol{x}^2 < 0 \tag{5.122}$$

which is always space-like. Hence, with this procedure we will get the correlation functions for space-like separations of its arguments. To get to time-like separations we will need to do an analytic continuation back to real time. This we will do later on. The second interpretation is that the path integral of Eq. (5.120) is the *partition function* of a system in Classical Statistical Mechanics in $D$ dimensions with energy density (divided by $T$) equal to $\mathcal{L}_E - J\phi$. This will turn out to be a very useful connection (both ways!).

### *5.5.2 Finite imaginary time interval*

In this case we have

$$0 \leq x_0 = \tau \leq \beta = 1/T \tag{5.123}$$

where $T$ will be interpreted as the temperature. Indeed, in this case the path integral is

$$Z'[0] = \mathrm{Tr}\, e^{-\beta H} \tag{5.124}$$

and we are effectively looking at a problem of the same Quantum Field Theory but at *finite temperature* $T = 1/\beta$. The path integral is once again the partition function but of a system in Quantum Statistical Physics! The partition function thus is (after setting $\hbar = 1$)

$$Z'[J] = \int \mathcal{D}\phi \; e^{-\int_0^\beta d\tau\, (\mathcal{L}_E - J\phi)} \tag{5.125}$$

where the field $\phi(\boldsymbol{x}, \tau)$ obeys *periodic boundary conditions* in imaginary time,

$$\phi(\boldsymbol{x}, \tau) = \phi(\boldsymbol{x}, \tau + \beta) \tag{5.126}$$

This boundary condition will hold for all *bosonic* theories. We will see later on that theories with fermions obey instead *anti-periodic* boundary conditions in imaginary time.

Hence, Quantum Field Theory at finite temperature $T$ is just Quantum Field Theory on an Euclidean space-time which is periodic (and finite) in one direction, imaginary time. In other words, we have wrapped (or *compactified*)

Euclidean space-time into a cylinder with perimeter (circumference) $\beta = 1/T$ (in units of $\hbar = k_B = 1$).

The correlation functions in imaginary time (which we will call the Euclidean correlation functions) are given by

$$\frac{1}{Z'[J]} \frac{\delta^N Z'[J]}{\delta J(x_1) \dots J(x_N)} \bigg|_{J=0} = \langle \phi(x_1) \dots \phi(x_N) \rangle \tag{5.127}$$

which are just the *correlation functions* in the equivalent problem in Statistical Mechanics. Upon analytic continuation the Euclidean correlation functions $\langle \phi(x_1) \dots \phi(x_N) \rangle$ and the $N$-point functions of the QFT are related by

$$\langle \phi(x_1) \dots \phi(x_N) \rangle \leftrightarrow (i\hbar)^N \langle 0 | T \phi(x_1) \dots \phi(x_N) | 0 \rangle \tag{5.128}$$

For the case of a quantum field theory at finite temperature $T$, the path integral yields the correlation functions of the Heisenberg field operators in imaginary time. These correlation functions are often called the *thermal correlation functions* (or propagators). They are functions of the spatial positions of the fields, $\boldsymbol{x}_1, \dots, \boldsymbol{x}_N$ and of their *imaginary time coordinates*, $x_{D1}, \dots, x_{DN}$ (here $x_D \equiv \tau$). To obtain the correlation functions as a function of the *real time coordinates* $x_{01}, \dots, x_{0N}$ at *finite temperature* $T$ it is necessary to do an analytic continuation. We will discuss how this is done later on.

## 5.6 Path integrals for the free scalar field

We will consider now the case of a *free scalar field*. We will carry our discussion in *Euclidean space-time* (i.e. in imaginary time), and we will do the relevant analytic continuation back to real time at the end of the calculation.

The Euclidean Lagrangian $\mathcal{L}_E$ for a free field $\phi$ coupled to a source $J$ is

$$\mathcal{L}_E = \frac{1}{2} \left(\partial_\mu \phi\right)^2 + \frac{1}{2} m^2 \phi^2 - J\phi \tag{5.129}$$

where we are using the notation

$$\left(\partial_\mu \phi\right)^2 = \partial_\mu \phi \partial_\mu \phi \tag{5.130}$$

Here the index is $\mu = 1, \dots, D$ for an Euclidean space-time of $D = d + 1$ dimensions. For the most part (but not always) we will be interested in the case of $d = 3$ and Euclidean space has four dimensions. Notice the way the Euclidean space-time indices are placed in Eq. (5.130). This is not a misprint!

We will compute the Euclidean Path Integral (or Partition Function) $\mathcal{Z}_E[J]$ exactly. The Euclidean Path Integral for a free field has the form

$$\mathcal{Z}_E[J] = \mathcal{N} \int \mathcal{D}\phi \; e^{-\int d^D x \left[\frac{1}{2}\left(\partial_\mu \phi\right)^2 + \frac{1}{2} m^2 \phi^2 - J\phi\right]} \tag{5.131}$$

In Classical Statistical Mechanics this theory is known as the Gaussian model.

In what follows I will assume that the boundary conditions of the field $\phi$ (and the source $J$) at infinity are either vanishing or periodic, and that the source $J$ also either vanishes at spatial infinity or is periodic. With these assumptions all terms which are total derivatives drop out identically. Therefore, upon an integration by parts and after dropping boundary terms, the Euclidean Lagrangian becomes

$$\mathcal{L}_E = \frac{1}{2}\phi \left[-\partial^2 + m^2\right]\phi - J\phi \tag{5.132}$$

Since this action is a quadratic form of the field $\phi$ this path integral can be calculated exactly. It has terms which are quadratic (or, rather bilinear) in $\phi$ and a term linear in $\phi$, the source term. By means of the following shift of the field $\phi$

$$\phi(x) = \bar{\phi}(x) + \xi(x) \tag{5.133}$$

the Lagrangian becomes

$$\begin{aligned} \mathcal{L}_E &= \frac{1}{2}\phi \left[-\partial^2 + m^2\right]\phi - J\phi \\ &= \frac{1}{2}\bar{\phi} \left[-\partial^2 + m^2\right]\bar{\phi} - J\bar{\phi} + \frac{1}{2}\xi \left[-\partial^2 + m^2\right]\xi + \xi \left[-\partial^2 + m^2\right]\bar{\phi} - J\xi \end{aligned} \tag{5.134}$$

Hence, we can decouple the source $J(x)$ by requiring that the shift $\bar{\phi}$ be such that the terms linear in $\xi$ cancel each other exactly. This requirement leads to the condition that the classical field $\bar{\phi}$ be the solution of the following inhomogeneous partial differential equation

$$\left[-\partial^2 + m^2\right]\bar{\phi} = J(x) \tag{5.135}$$

Equivalently, we can write the classical field $\bar{\phi}$ is terms of the source $J(x)$ through the action of the inverse of the operator $-\partial^2 + m^2$,

$$\bar{\phi} = \frac{1}{-\partial^2 + m^2}\, J \tag{5.136}$$

The solution of Eq. (5.135) is

$$\bar{\phi}(x) = \int d^D x' \, G_0^E(x - x') \, J(x') \tag{5.137}$$

where

$$G_0^E(x - x') = \langle x | \frac{1}{-\partial^2 + m^2} | x' \rangle \tag{5.138}$$

is the correlation function of the linear partial differential operator $-\partial^2 + m^2$. Thus, $G_0^E(x - x')$ is the solution of

$$\left[-\partial_x^2 + m^2\right] G_0^E(x - x') = \delta^D(x - x') \tag{5.139}$$

In terms of $G_0^E(x - x')$, the terms of the shifted action become,

$$\begin{aligned}
\int d^D x \Big( \frac{1}{2}\bar{\phi}(x) \, &\left[-\partial^2 + m^2\right]\bar{\phi}(x) - J\bar{\phi}(x) \Big) \\
&= -\frac{1}{2} \int d^D x \bar{\phi}(x) J(x) \\
&= -\frac{1}{2} \int d^D x \, d^D x' \, J(x) \, G_0^E(x - x') \, J(x')
\end{aligned} \tag{5.140}$$

Therefore the path integral for the generating function of the free Euclidean scalar field $Z_E[J]$, defined in Eq. (5.131), is given by

$$Z_E[J] = Z_E[0] \; e^{\frac{1}{2} \int d^D x \int d^D x' \, J(x) \, G_0^E(x - x') \, J(x')} \tag{5.141}$$

where $Z_E[0]$ is

$$Z_E[0] = \int \mathcal{D}\xi \; e^{-\frac{1}{2} \int d^D x \, \xi(x) \left[-\partial^2 + m^2\right] \xi(x)} \tag{5.142}$$

Eq. (5.141) shows that, after the decoupling, $Z_E[J]$ is a product of two factors: (a) a factor that is function of a bilinear form in the source $J$, and (b) a path integral, $Z_E[0]$, that is independent of the sources.

### 5.6.1 Calculation of $Z_E[0]$

The path integral $Z_E[0]$ is analogous to the fluctuation factor that we found in the path integral for a harmonic oscillator in elementary quantum mechanics. There we saw that the analogous factor could be written as a determinant of a differential operator, the kernel of the bilinear form that entered in the action. The same result holds here as well. The only difference is that the kernel is now the *partial differential* operator $\hat{A} = -\partial^2 + m^2$ whereas in Quantum Mechanics is an ordinary differential operator. Here too, the operator $\hat{A}$ has a set of eigenstates $\{\Psi_n(x)\}$ which, once the boundary conditions in space-time are specified, are both complete and orthonormal, and the associated spectrum of eigenvalues $A_n$ is

$$\begin{aligned}\left[-\partial^2 + m^2\right]\Psi_n(x) =& A_n \Psi_n(x)\\ \int d^D x\ \Psi_n(x)\ \Psi_m(x) =& \delta_{n,m}\\ \sum_n \Psi_n(x)\ \Psi_n(x') =& \delta(x - x')\end{aligned} \tag{5.143}$$

Hence, once again we can expand the field $\phi(x)$ in the complete set of states $\{\Psi_n(x)\}$

$$\phi(x) = \sum_n c_n\ \Psi_n(x) \tag{5.144}$$

Hence, the field configurations are thus parametrized by the coefficients $\{c_n\}$.

The action now becomes,

$$S = \int d^D x\ \mathcal{L}_E(\phi, \partial\phi) = \frac{1}{2}\sum_n A_n c_n^2 \tag{5.145}$$

Thus, up to a normalization factor, we find that $Z_E[0]$ is given by

$$Z_E[0] = \prod_n A_n^{-1/2} \equiv \left(\operatorname{Det}\left[-\partial^2 + m^2\right]\right)^{-1/2}, \tag{5.146}$$

and we have reduced the calculation of $Z_E[0]$ to the computation of the determinant of a differential operator, $\operatorname{Det}\left[-\partial^2 + m^2\right]$.

In chapter 8 we will discuss efficient methods to compute such determinants. For the moment, it will be sufficient to notice that there is a simple, but formal, way to compute this determinant. First, we notice that if we are interested in the behavior of an infinite system at $T = 0$, the eigenstates of the operator $-\partial^2 + m^2$ are simply suitably normalized plane waves. Let $L$ be the linear size of the system, with $L \to \infty$. Then, the eigenfunctions are labeled by a $D$-dimensional momentum $p_\mu$ (with $\mu = 0, 1, \ldots, d$)

$$\Psi_p(x) = \frac{1}{(2\pi L)^{D/2}} e^{i\, p_\mu\, x_\mu} \tag{5.147}$$

with eigenvalues,

$$A_p = p^2 + m^2 \tag{5.148}$$

Hence the *logarithm* of determinant is

$$\begin{aligned}\ln \mathrm{Det}\left[-\partial^2+m^2\right] =& \mathrm{Tr} \ln\left[-\partial^2+m^2\right]\\ =&\sum_p \ln(p^2+m^2)\\ =&V\int \frac{d^Dp}{(2\pi)^D}\ln(p^2+m^2)\end{aligned} \tag{5.149}$$

where $V=L^D$ is the volume of Euclidean space-time. Hence,

$$\ln Z_E[0]=-\frac{V}{2}\int \frac{d^Dp}{(2\pi)^D}\ln(p^2+m^2) \tag{5.150}$$

This expression is has two singularities: an *infrared divergence* and an *ultraviolet divergence*. $\ln Z[0]$, Eq(5.150), diverges as $V \to \infty$. This *infrared* (IR) singularity actually is not a problem since $\ln Z_E[0]$ should be an *extensive* quantity which must scale with the volume of space-time. In other words, this is how it should behave. However, the integral in Eq.(5.150) diverges at large momenta unless there is an upper bound (or *cutoff*) for the allowed momenta. This is an *ultraviolet* (UV) singularity. It has the same origin of the UV divergence of the ground state energy. In fact $Z_E[0]$ is closely related to the ground state (vacuum) energy since

$$Z_E[0]=\lim_{\beta\to\infty}\sum_n e^{-\beta E_n}\sim e^{-\beta E_0}+\dots \tag{5.151}$$

Thus,

$$E_0=-\lim_{\beta\to\infty}\frac{1}{\beta}\ln Z_E[0]=\frac{1}{2}L^d\int \frac{d^Dp}{(2\pi)^D}\ln(p^2+m^2) \tag{5.152}$$

where $L^d$ is the volume of *space*, and $V=L^d\beta$. Notice that Eq. (5.152) is UV divergent. Later in this chapter we will discuss how to compute expressions of the form of Eq. (5.152).

### 5.6.2 Propagators and correlators

A number of interesting results are found immediately by direct inspection of Eq. (5.141). We can easily see that, once we set $J=0$, the correlation function $G_E^{(0)}(x-x')$

$$G_E^{(0)}(x-x')=\langle x|\frac{1}{-\partial^2+m^2}|x'\rangle \tag{5.153}$$

is equal to the 2-point correlation function for this theory (at $J = 0$),

$$\langle \phi(x)\phi(x')\rangle = \frac{1}{Z_E[0]} \frac{\delta^2 Z_E[J]}{\delta J(x)\delta J(x')}\bigg|_{J=0} = G_E^{(0)}(x-x') \tag{5.154}$$

Likewise we find that, for a free field theory, the $N$-point correlation function $\langle \phi(x_1)\dots\phi(x_N)\rangle$ is equal to

$$\begin{aligned}\langle \phi(x_1)\dots\phi(x_N)\rangle =& \frac{1}{Z_E[0]} \frac{\delta^N Z_E[J]}{\delta J(x_1)\dots\delta J(x_N)}\bigg|_{J=0} \\ =& \langle \phi(x_1)\phi(x_2)\rangle \dots \langle \phi(x_{N-1})\phi(x_N)\rangle + \text{permutations}\end{aligned} \tag{5.155}$$

Therefore, for a free field, up to permutations of the coordinates $x_1, \dots, x_N$, the $N$-point functions reduces to a sum of products of 2-point functions. Hence, $N$ must be a positive *even* integer. This result, Eq. (5.155), which we derived in the context of a theory for a *free scalar* field, is actually much more general. It is known as *Wick's Theorem*. It applies to all free theories, theories whose Lagrangians are bilinear in the fields, it is independent of the statistics and on whether there is relativistic invariance or not. The only caveat is that, as we will see later on, for the case of fermionic theories there is a sign associated with each term of this sum.

It is easy to see that, for $N = 2k$, the total number of terms in the sum is

$$(2k-1)(2k-3)\dots\dots = \frac{(2k)!}{2^k k!} \tag{5.156}$$

Each factor of a 2-point function $\langle \phi(x_1)\phi(x_2)\rangle$, a free propagator, and it is also called a *contraction*. It also common to use the notation

$$\langle \phi(x_1)\phi(x_2)\rangle = \overbracket{\phi(x_1)\phi}(x_2) \tag{5.157}$$

to denote a contraction or propagator.

### 5.6.3 Calculation of the propagator

We will now calculate the 2-point function, or propagator, $G_E^{(0)}(x-x')$ for infinite Euclidean space. This is the case of interest in QFT at $T = 0$. Later on we will do the calculation of the propagator at finite temperature.

Eq. (5.139) tells us that $G_E^{(0)}(x-x')$ is the Green function of the operator $-\partial^2 + m^2$. We will use Fourier transform methods and write $G_E^{(0)}(x-x')$ in the form

$$G_E^{(0)}(x-x') = \int \frac{d^D p}{(2\pi)^D}\, G_0^E(p)\, e^{i\,p_\mu (x_\mu - x'_\mu)} \tag{5.158}$$

which is a solution of Eq. (5.139) if

$$G_E^{(0)}(p) = \frac{1}{p^2+m^2} \tag{5.159}$$

Therefore the correlation function in real (Euclidean!) space is the integral

$$G_E^{(0)}(x-x') = \int \frac{d^D p}{(2\pi)^D}\, \frac{e^{i\,p_\mu (x_\mu - x'_\mu)}}{p^2+m^2} \tag{5.160}$$

We will often encounter integrals of this type and for that reason we will do this one in some detail. We begin by using the identity

$$\frac{1}{A} = \frac{1}{2}\int_0^\infty d\alpha\, e^{-\frac{A}{2}\alpha} \tag{5.161}$$

where $A > 0$ is a positive real number. The variable $\alpha$ is called a Feynman-Schwinger parameter.

We now choose $A = p^2 + m^2$, and substitute this expression back in Eq. (5.160), which takes the form

$$G_E^{(0)}(x-x') = \frac{1}{2}\int_0^\infty d\alpha \int \frac{d^D p}{(2\pi)^D}\, e^{-\frac{\alpha}{2}(p^2+m^2) + ip_\mu(x_\mu - x'_\mu)} \tag{5.162}$$

The integrand is a Gaussian, and the integral can be calculated by a shift of the integration variables $p_\mu$, i.e. by completing squares

$$\frac{\alpha}{2}(p^2+m^2) - ip_\mu(x_\mu - x'_\mu) = \frac{1}{2}\left(\sqrt{\alpha}p_\mu - i\frac{x_\mu - x'_\mu}{\sqrt{\alpha}}\right)^2 - \frac{1}{2}\left(\frac{x_\mu - x'_\mu}{\sqrt{\alpha}}\right)^2 \tag{5.163}$$

and by using the Gaussian integral

$$\int \frac{d^D p}{(2\pi)^D}\, e^{-\frac{1}{2}\left(\sqrt{\alpha}p_\mu - i\frac{x_\mu - x'_\mu}{\sqrt{\alpha}}\right)^2} = (2\pi\alpha)^{-D/2} \tag{5.164}$$

After all of this is done, we find the formula

$$G_E^{(0)}(x-x') = \frac{1}{2(2\pi)^{D/2}} \int_0^\infty d\alpha\, \alpha^{-D/2}\, e^{-\frac{|x-x'|^2}{2\alpha} - \frac{1}{2}m^2\alpha} \tag{5.165}$$

Let us now define a rescaling of the variable $\alpha$,

$$\alpha = \lambda t \tag{5.166}$$

by which

$$\frac{|x-x'|^2}{2\alpha} + \frac{1}{2}m^2\alpha = \frac{|x-x'|^2}{2\lambda t} + \frac{1}{2}m^2\lambda t \tag{5.167}$$

If we choose

$$\lambda = \frac{|x-x'|}{m}, \tag{5.168}$$

the exponent becomes

$$\frac{|x-x'|^2}{2\alpha} + \frac{1}{2}m^2\alpha = \frac{m|x-x'|}{2}\left(t + \frac{1}{t}\right) \tag{5.169}$$

After this final change of variables, we find that the correlation function is

$$G_E^{(0)}(x-x') = \frac{1}{(2\pi)^{D/2}}\left(\frac{m}{|x-x'|}\right)^{\frac{D}{2}-1} K_{\frac{D}{2}-1}(m|x-x'|) \tag{5.170}$$

where $K_\nu(z)$ is the Modified Bessel function, which has the integral representation

$$K_\nu(z) = \frac{1}{2}\int_0^\infty dt\, t^{\nu-1}\, e^{-\frac{z}{2}\left(t+\frac{1}{t}\right)} \tag{5.171}$$

where $\nu = \frac{D}{2} - 1$, and $z = m|x-x'|$.

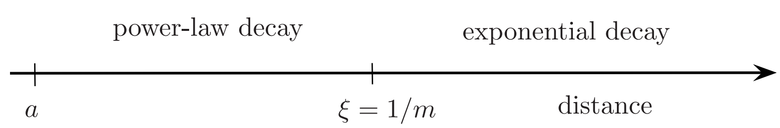


Figure 5.6 Behaviors of the Euclidean propagator.

There are two interesting regimes: (a) long distances, $m|x-x'| \gg 1$ , and (b) short distances, $m|x-x'| \ll 1$.

*A: long distance behavior*

In this regime, $z = m|x-x'| \gg 1$, a saddle-point calculation shows that the Bessel Function $K_\nu(z)$ has the asymptotic behavior,

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}}e^{-z}\left[1 + O(1/z)\right] \tag{5.172}$$

Thus, in this regime the Euclidean propagator (or correlation function) behaves like

$$G_E^{(0)}(x-x') = \frac{\sqrt{\pi/2}\, m^{D-2}\, e^{-m|x-x'|}}{(2\pi)^{D/2}\,(m|x-x'|)^{\frac{D-1}{2}}} \left[1+O\left(\frac{1}{m|x-x'|}\right)\right] \tag{5.173}$$

Therefore, at long distances, the Euclidean (or imaginary time) propagator has an exponential decay with distance (and imaginary time). The length scale for this decay is $1/m$, which is natural since it is the only quantity with units of length in the theory. In real time, and in conventional units, this length scale is just the Compton wavelength, $\hbar/mc$. In Statistical Physics this length scale is known as the *correlation length* $\xi$.

*B: short distance behavior*

In this regime we must use the behavior of the Bessel function for small values of the argument,

$$K_\nu(z) = \frac{\Gamma(\nu)}{2\left(\frac{z}{2}\right)^\nu} + O(1/z^{\nu-2}) \tag{5.174}$$

The correlation function now behaves instead like,

$$G_E^{(0)}(x-x') = \frac{\Gamma(\frac{D}{2}-1)}{4\pi^{D/2}|x-x'|^{D-2}} + \dots \tag{5.175}$$

where $\dots$ are terms that vanish as $m|x-x'| \to 0$. Notice that the leading term is *independent of the mass m*. This is the behavior of the free *massless* theory.

### 5.6.4 Behavior of the propagator in Minkowski space-time

We will now find the behavior of the propagator in *real time*. This means that now we must do the analytic continuation back to real time.

Let us recall that in going from Minkowski to Euclidean space we continued $x_0 \to -ix_4$. In addition, there is also factor of $i$ difference in the definition of the propagator. Thus, the propagator in Minkowski space-time $G^{(0)}(x-x')$ is the expression that results from the analytic continaution

$$G^{(0)}(x-x') = iG_E^{(0)}(x-x')\big|_{x_4\to ix_0} \tag{5.176}$$

We can also obtain this result from the path integral formulation in

Minkowski space-time. Indeed, the generating functional for a free real massive scalar field $Z[J]$ in $D = d + 1$ dimensional Minkowski space-time is

$$Z[J] = \int \mathcal{D}\phi e^{i \int d^D x \left[ \frac{1}{2} (\partial\phi)^2 - \frac{m^2}{2}\phi^2 + J\phi \right]} \tag{5.177}$$

Hence, the expectation value to the time-ordered product of two field is

$$\langle 0|T\phi(x)\phi(y)|0\rangle = -\frac{1}{Z[J]} \frac{\delta Z[J]}{\delta J(x)\delta J(y)}\Big|_{J=0} \tag{5.178}$$

On the other hand, for a free field the generating function is given by (up to a normalization constant $\mathcal{N}$)

$$Z[J] = \mathcal{N}\left[\text{Det}\left(\partial^2 + m^2\right)\right]^{-1/2} e^{\frac{i}{2}\int d^D x \int d^D y J(x) G^{(0)}(x-y) J(y)} \tag{5.179}$$

where $G_0(x - y)$ is the Green function of the Klein-Gordon operator and satisfies

$$\left(\partial^2 + m^2\right) G^{(0)}(x-y) = \delta^D(x-y) \tag{5.180}$$

Hence, we obtain the expected result

$$\langle 0|T\phi(x)\phi(y)|0\rangle = -iG^{(0)}(x-y) \tag{5.181}$$

Let us compute the propagator in $D = 4$ Minkowski space-time by analytic continuation from the $D = 4$ Euclidean propagator. The relativistic interval $s$ is given by

$$s^2 = (x_0 - x_0')^2 - (\boldsymbol{x} - \boldsymbol{x}')^2 \tag{5.182}$$

The Euclidean interval (length) $|x - x'|$, and the relativistic interval $s$ are related by

$$|x - x'| = \sqrt{(x - x')^2} \to \sqrt{-s^2} \tag{5.183}$$

Therefore, in $D = 4$ space-time dimensions, the Minkowski space propagator is

$$G^{(0)}(x - x') = \frac{i}{4\pi^2} \frac{m}{\sqrt{-s^2}} K_1(m\sqrt{-s^2}) \tag{5.184}$$

We will need the asymptotic behavior of the Bessel function $K_1(z)$,

$$K_1(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left[1 + \frac{3}{8z} + \dots\right] \qquad , \quad \text{for} \quad z \gg 1$$
$$K_1(z) = \frac{1}{z} + \frac{z}{2}\left(\ln z + C - \frac{1}{2}\right) + \dots \qquad , \quad \text{for} \quad z \ll 1 \tag{5.185}$$

where $C = 0.577215\dots$ is the Euler-Mascheroni constant. Let us examine

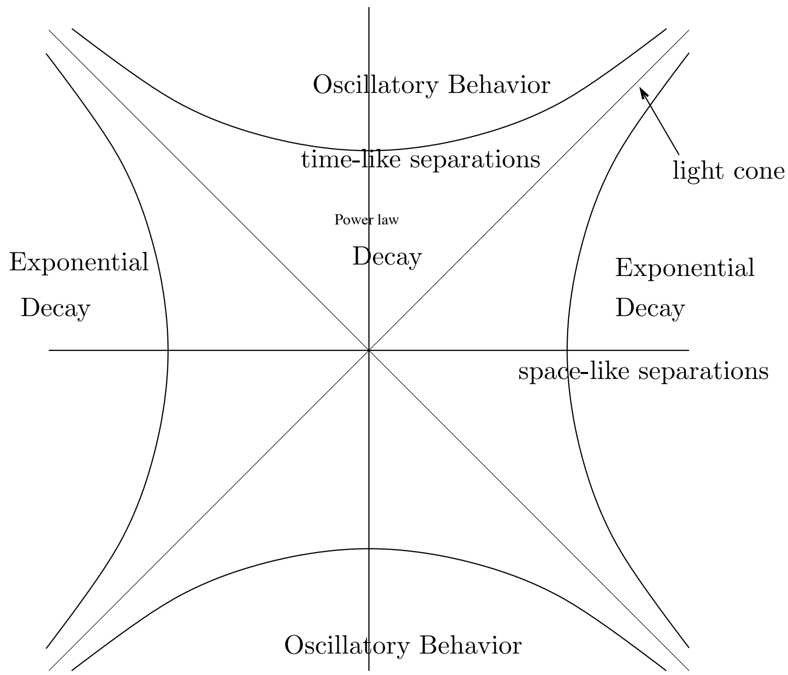


Figure 5.7 Behaviors of the propagator in Minkowski space-time.

the behavior of Eq. (5.184) in the regimes: (a) space-like, $s^2 < 0$, and (b) time-like, $s^2 > 0$, intervals.

*A: space-like intervals:* $(x - x')^2 = s^2 < 0$

This is the space-like domain. By inspecting Eq. (5.184) we see that for space-like separations, the factor $\sqrt{-s^2}$ is a positive real number. Consequently the argument of the Bessel function is real (and positive), and the propagator is pure imaginary. In particular we see that, for $s^2 < 0$ the Minkowski propagator is essentially the Euclidean correlation function,

$$G^{(0)}(x - x') = iG_E^{(0)}(x - x'), \qquad \text{for} \quad s^2 < 0 \tag{5.186}$$

Hence, for $s^2 < 0$ we have the asymptotic behaviors,

$$G^{(0)}(x-x') = i\frac{\sqrt{\pi/2}}{4\pi^2}\frac{m^2}{\left(m\sqrt{-s^2}\right)^{3/2}}e^{-m\sqrt{-s^2}}, \qquad \text{for } m\sqrt{-s^2} \gg 1$$

$$G^{(0)}(x-x') = \frac{i}{4\pi^2(-s^2)}, \qquad \text{for } m\sqrt{-s^2} \ll 1 \quad (5.187)$$

*B: time-like intervals:* $(x-x')^2 = s^2 > 0$

This is the time-like domain. The analytic continuation yields

$$G^{(0)}(x-x') = \frac{m}{4\pi^2\sqrt{s^2}}K_1(im\sqrt{s^2}) \qquad (5.188)$$

For pure imaginary arguments, the Bessel function $K_1(iz)$ is the analytic continuation of the Hankel function, $K_1(iz) = -\frac{\pi}{2}H_1^{(1)}(-z)$ . This function is oscillatory for large values of its argument. Indeed, we now get the behaviors

$$G^{(0)}(x-x') = \frac{\sqrt{\pi/2}}{4\pi^2}\frac{m^2}{\left(m\sqrt{s^2}\right)^{3/2}}e^{im\sqrt{s^2}}, \qquad \text{for } m\sqrt{s^2} \gg 1$$

$$G^{(0)}(x-x') = \frac{1}{4\pi^2 s^2}, \qquad \text{for } m\sqrt{s^2} \ll 1 \quad (5.189)$$

Notice that, up to a factor of $i$, the short distance behavior is the same for both time-like and space like separations. The main difference is that at large time-like separations we get an oscillatory behavior instead of an exponential decay. The length scale of the oscillations is, once again, set by the only scale in the theory, the Compton wavelength.

## 5.7 Exponential decays and mass gaps

The exponential decay at long space-like separations (and the oscillatory behavior at long time-like separations) is not a peculiarity of the free field theory. It is a general consequence of the existence of a *mass gap* in the spectrum. We can see that by considering the 2-point function of a *generic* theory, for simplicity in imaginary time. The 2-point function is

$$G^{(2)}(\boldsymbol{x}-\boldsymbol{x}', \tau-\tau') = \langle 0|T\hat{\phi}(\boldsymbol{x},\tau)\hat{\phi}(\boldsymbol{x}',\tau')|0\rangle \qquad (5.190)$$

where $T$ is the imaginary time-ordering operator.

The Heisenberg representation of the operator $\hat{\phi}$ in imaginary time is ($\hbar = 1$)

$$\hat{\phi}(\boldsymbol{x},\tau) = e^{H\tau}\,\hat{\phi}(\boldsymbol{x},0)\,e^{-H\tau} \qquad (5.191)$$

Hence, we can write the 2-point function as

$$
\begin{aligned}
G^{(2)}(\boldsymbol{x}-\boldsymbol{x}',\tau-\tau') =& \\
&= \theta(\tau-\tau')\langle 0|e^{H\tau}\ \hat{\phi}(\boldsymbol{x},0)\ e^{-H(\tau-\tau')}\ \hat{\phi}(\boldsymbol{x}',0)\ e^{-H\tau'}|0\rangle \\
&+ \theta(\tau'-\tau)\langle 0|e^{H\tau'}\ \hat{\phi}(\boldsymbol{x}',0)\ e^{-H(\tau'-\tau)}\ \hat{\phi}(\boldsymbol{x},0)\ e^{-H\tau}|0\rangle \\
&= \theta(\tau-\tau')\ e^{E_0(\tau-\tau')}\ \langle 0|\hat{\phi}(\boldsymbol{x},0)\ e^{-H(\tau-\tau')}\ \hat{\phi}(\boldsymbol{x}',0)|0\rangle \\
&+ \theta(\tau'-\tau)\ e^{E_0(\tau'-\tau)}\ \langle 0|\hat{\phi}(\boldsymbol{x}',0)\ e^{-H(\tau'-\tau)}\ \hat{\phi}(\boldsymbol{x},0)|0\rangle
\end{aligned}
\tag{5.192}
$$

We now insert a complete set of eigenstates $\{|n\rangle\}$ of the Hamiltonian $\hat{H}$, with eigenvalues $\{E_n\}$. The 2-point function now reads,

$$
\begin{aligned}
G^{(2)}(\boldsymbol{x}-\boldsymbol{x}',\tau-\tau') =& \\
&= \theta(\tau-\tau')\ \sum_n\ \langle 0|\hat{\phi}(\boldsymbol{x},0)|n\rangle\ \langle n|\hat{\phi}(\boldsymbol{x}',0)|0\rangle\ e^{-(E_n-E_0)(\tau-\tau')} \\
&+ \theta(\tau'-\tau)\ \sum_n\ \langle 0|\hat{\phi}(\boldsymbol{x}',0)|n\rangle\ \langle n|\hat{\phi}(\boldsymbol{x},0)|0\rangle\ e^{-(E_n-E_0)(\tau'-\tau)}
\end{aligned}
\tag{5.193}
$$

Since

$$
\hat{\phi}(\boldsymbol{x},0) = e^{i\hat{\boldsymbol{P}}\cdot\boldsymbol{x}}\hat{\phi}(0,0)e^{-i\hat{\boldsymbol{P}}\cdot\boldsymbol{x}}
\tag{5.194}
$$

and, in a translation invariant system, the eigenstates of the Hamiltonian are also eigenstates of the total momentum $\boldsymbol{P}$

$$
\hat{\boldsymbol{P}}|0\rangle = 0, \qquad \hat{\boldsymbol{P}}|n\rangle = \boldsymbol{P}_n|n\rangle,
\tag{5.195}
$$

where $\boldsymbol{P}_n$ is the linear momentum of state $|n\rangle$, we can write

$$
\langle 0|\hat{\phi}(\boldsymbol{x},0)|n\rangle\ \langle n|\hat{\phi}(\boldsymbol{x}',0)|0\rangle = |\langle 0|\hat{\phi}(0,0)|n\rangle|^2\ e^{-i\boldsymbol{P}_n\cdot(\boldsymbol{x}-\boldsymbol{x}')}
\tag{5.196}
$$

Using the above expressions we can rewrite Eq. (5.193) in the form

$$
\begin{aligned}
&G^{(2)}(\boldsymbol{x}-\boldsymbol{x}',\tau-\tau') \\
&= \sum_n |\langle 0|\hat{\phi}(0,0)|n\rangle|^2 \Big[\theta(\tau-\tau')e^{-i\boldsymbol{P}_n\cdot(\boldsymbol{x}-\boldsymbol{x}')}e^{-(E_n-E_0)(\tau-\tau')} \\
&\qquad\qquad + \theta(\tau'-\tau)e^{-i\boldsymbol{P}_n\cdot(\boldsymbol{x}'-\boldsymbol{x})}e^{-(E_n-E_0)(\tau'-\tau)}\Big]
\end{aligned}
\tag{5.197}
$$

Thus, at equal positions, $\boldsymbol{x} = \boldsymbol{x}'$, we obtain the following simpler expression in the imaginary time interval $\tau - \tau'$

$$G^{(2)}(0, \tau - \tau') = \sum_n |\langle 0|\hat{\phi}(\mathbf{0}, 0)|n\rangle|^2 \times e^{-(E_n - E_0)|\tau - \tau'|} \tag{5.198}$$

In the limit of large imaginary time separation, $|\tau - \tau'| \to \infty$, there is always a largest non-vanishing term in the sums. This is the term for the state $|n_0\rangle$ that mixes with the vacuum state $|0\rangle$ through the field operator $\hat{\phi}$, and with the *lowest* excitation energy, the *mass gap* $E_{n_0} - E_0$. Hence, for large imaginary time separations, $|\tau - \tau'| \to \infty$, the 2-point function decays exponentially,

$$G^{(2)}(0, \tau - \tau') \simeq |\langle 0|\hat{\phi}(\mathbf{0}, 0)|n_0\rangle|^2 \times e^{-(E_{n_0} - E_0)|\tau - \tau'|} \tag{5.199}$$

a result that we already derived for a free field in Eq.(5.173). Therefore, if the spectrum has a gap, the correlation functions (or propagators) decay exponentially in imaginary time. In real time we will get, instead, an oscillatory behavior. This is a very general result.

Finally, notice that Lorentz invariance in Minkowski space-time (real time) implies rotational (Euclidean) invariance in imaginary time. Hence, exponential decay in imaginary times, at equal positions, must imply (in general) exponential decay in real space at equal imaginary times. Thus, in a Lorentz invariant system the propagator at space-like separations is always equal to the propagator in imaginary time.

## 5.8 Scalar fields at finite temperature

We will now discuss briefly the behavior of free scalar fields in thermal equilibrium at finite temperature $T$. We will give a more detailed discussion in Chapter 10 where we will discuss more extensively the relation between observables and propagators.

We saw in Section 5.5 that the field theory is now defined on an Euclidean cylindrical space-time which is finite and periodic along the imaginary time direction with circumference $\beta = 1/T$, where $T$ is the temperature (where we set the Boltzmann constant $k_B = 1$). Hence the imaginary time dimension has been compactified.

### 5.8.1 The free energy

Let us begin by computing the free energy. We will work in $D = d + 1$ Euclidean space-time dimensions. The partition function $Z(T)$ is computed by the result of Eq.(5.146) except that the differential operator now is

$$\hat{A} = -\partial_\tau^2 - \partial^2 + m^2, \tag{5.200}$$

with the caveat that now $\partial^2$ denotes the the Laplacian operator that acts only on the spacial coordinates, $\mathbf{x}$, and that the imaginary time is periodic. The mode expansion for the field in this Euclidean (cylinder) space is

$$\phi(\boldsymbol{x},\tau) = \sum_{n=-\infty}^{\infty} \int \frac{d^d p}{(2\pi)^d} \phi(\omega_n, \boldsymbol{p}) e^{i\omega_n \tau + i\boldsymbol{p}\cdot\boldsymbol{x}} \tag{5.201}$$

where $\omega_n = 2\pi T n$ are the Matsubara frequencies and $n \in \mathbb{Z}$. The field operator is periodic in the imaginary time $\tau$ with period $\beta = 1/T$. The Euclidean action now is

$$\begin{aligned} S =& \int_0^\beta d\tau \int d^d x \left[ \frac{1}{2} (\partial_\tau \phi)^2 + \frac{1}{2} (\partial \phi)^2 + \frac{1}{2} m^2 \phi^2 \right] \\ =& \frac{\beta}{2} \int \frac{d^d p}{(2\pi)^d} (\boldsymbol{p}^2 + m^2) |\phi_0(\boldsymbol{p})|^2 \\ &+ \beta \int \frac{d^d p}{(2\pi)^d} \sum_{n \geq 1} \left( \omega_n^2 + \boldsymbol{p}^2 + m^2 \right) |\phi(\omega_n, \boldsymbol{p})|^2 \end{aligned} \tag{5.202}$$

where we split the action into the sum of the contribution from the zero-frequency Matsubara mode, denoted by $\phi_0(\boldsymbol{p}) = \phi(0, \boldsymbol{p})$, and the contributions of the modes for the rest of the frequencies.

Since the free energy is given by $F(T) = -T \ln Z(T)$, we need to compute (again, up to the usual UV divergent normalization constant)

$$F(T) = \frac{T}{2} \ln \operatorname{Det} \big[ -\partial_\tau^2 - \partial^2 + m^2 \big] \tag{5.203}$$

We can now expand the determinant in the eigenvalues of the operator $-\partial_\tau^2 - \partial^2 + m^2$, and obtain the formally divergent expression

$$F(T) = \frac{1}{2} V T \int \frac{d^d p}{(2\pi)^d} \sum_{n=-\infty}^{\infty} \ln \left( \beta \big[ \omega_n^2 + \boldsymbol{p}^2 + m^2 \big] \right) \tag{5.204}$$

where $V$ is the spatial volume. This expression is formally divergent both in the momentum integrals and in the frequency sum and needs to be regularized. We already encountered this problem in our discussion of path integrals in Quantum Mechanics. As in that case we will recall that we have a formally divergent normalization constant, $\mathcal{N}$, which we have not made explicit here and that can be defined as to cancel the divergence of the frequency sum (as we did in Eq.(5.84)).

The regularized frequency sum can now be computed

$$F(T) = VT \int \frac{d^d p}{(2\pi)^d} \ln\left[\left(\beta\left(\boldsymbol{p}^2 + m^2\right)^{1/2}\right)\prod_{n=1}^{\infty}\left(1 + \frac{\boldsymbol{p}^2 + m^2}{\omega_n^2}\right)\right] \tag{5.205}$$

Using the identity of Eq.(5.85) the free energy $F(T)$ becomes

$$F(T) = VT \int \frac{d^d p}{(2\pi)^d} \ln\left[2\sinh\left(\frac{\sqrt{\boldsymbol{p}^2 + m^2}}{2T}\right)\right] \tag{5.206}$$

which can be written in the form

$$F(T) = V\varepsilon_0 + VT \int \frac{d^d p}{(2\pi)^d} \ln\left(1 - e^{-\frac{\sqrt{\boldsymbol{p}^2 + m^2}}{T}}\right) \tag{5.207}$$

where

$$\varepsilon_0 = \frac{1}{2}\int \frac{d^d p}{(2\pi)^d}\sqrt{\boldsymbol{p}^2 + m^2} \tag{5.208}$$

is the (ultraviolet divergent) vacuum (ground state) energy density. Notice that the ultraviolet divergence is absent in the finite temperature contribution.

### 5.8.2 The thermal propagator

The thermal propagator is the time-ordered propagator in imaginary time. It is equivalent to the Euclidean correlation function on the cylindrical geometry. We will denote the thermal propagator by

$$G_T^{(0)}(\boldsymbol{x}, \tau) = \langle \phi(\boldsymbol{x}, \tau)\phi(\boldsymbol{0}, 0)\rangle_T \tag{5.209}$$

It has the Fourier expansion

$$\langle \phi(\boldsymbol{x}, \tau)\phi(\boldsymbol{0}, 0)\rangle_T = \frac{1}{\beta}\sum_{n=-\infty}^{\infty}\int \frac{d^d p}{(2\pi)^d}\frac{e^{i\omega_n\tau + i\boldsymbol{p}\cdot\boldsymbol{x}}}{\omega_n^2 + \boldsymbol{p}^2 + m^2} \tag{5.210}$$

where, once again, $\omega_n = 2\pi T n$ are the Matsubara frequencies.

We will now obtain two useful expressions for the thermal propagator. The expression follows from doing the momentum integrals first. In fact, by observing that the Matsubara frequencies act as mass terms of a field in one dimension lower, which allows us to identify the integrals in Eq.(5.210) with the Euclidean propagators of an infinite number of fields, each labeled by an integer $n$, in $d$ Euclidean dimensions with mass squared equal to

$$m_n^2 = m^2 + \omega_n^2 \tag{5.211}$$

We can now use the result of Eq.(5.170) for the Euclidean correlator (now in $d$ Euclidean dimensions) and write the thermal propagator as the following series

$$G_T^{(0)}(\boldsymbol{x}, \tau) = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \frac{e^{i\omega_n \tau}}{(2\pi)^{d/2}} \left( \frac{m_n}{|x - x'|} \right)^{\frac{d}{2} - 1} K_{\frac{d}{2}-1}(m_n |\boldsymbol{x}|) \tag{5.212}$$

where $m_n$ is given in Eq.(5.211). Since the thermal propagator is expressed as an infinite series of massive propagators, each with increasing masses, it implies that at distances large compared with the length scale $\lambda_T = (2\pi T)^{-1}$, known as the thermal wavelength, all the terms of the series become negligible compared with the term with vanishing Matsubara frequency. In this limit, the thermal propagator reduces to the correlator of the classical theory in $d$ (spatial) Euclidean dimensions,

$$G_T^{(0)}(\boldsymbol{x}, \tau) \simeq \langle \phi(\boldsymbol{x}) \phi(\boldsymbol{0}) \rangle, \qquad \text{for } |\boldsymbol{x}| \gg \lambda_T \tag{5.213}$$

In other terms, at distances large compared with the circumference $\beta$ of the the cylindrical Euclidean space-time, the theory becomes asymptotically equivalent to the Euclidean theory in one space-time dimension less.

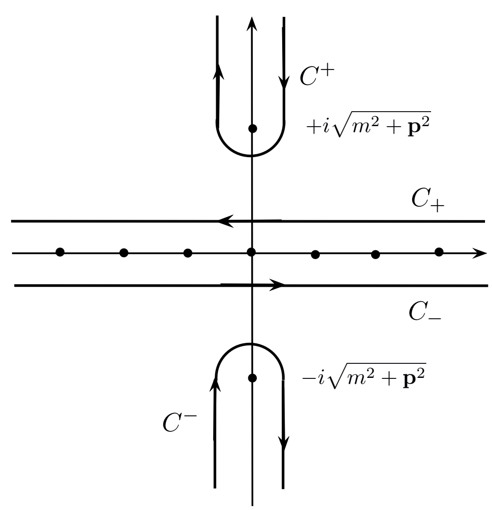


Figure 5.8

We will now find an alternative expression for the thermal propagator by doing the sum over Matsubara frequencies shown in Eq.(5.210). We will use the residue theorem to represent the sum as a contour integral on the complex plane, as shown in Fig.5.8,

$$\frac{1}{\beta}\sum_{n=-\infty}^{\infty}\frac{e^{i\omega_n\tau}}{\omega_n^2+\boldsymbol{p}^2+m^2}=\frac{1}{2}\oint_{C_+\cup C_-}\frac{dz}{2\pi i}\frac{e^{iz\tau}}{z^2+m^2+\boldsymbol{p}^2}\cot\left(\frac{z}{2T}\right) \tag{5.214}$$

where the (positively oriented) contour $C = C_- \cup C_+$ in the complex $z$ plane is shown in Fig.5.8. The black dots on the real axis of represent the integers $z = n$, while the black dots on the imaginary axis represent the poles at $\pm i\sqrt{m^2+\boldsymbol{p}^2}$. Upon distorting the contour $C_+$ to the negatively oriented contour $C^+$ of the upper half-plane, and the contour $C_-$ to the negatively oriented contour $C^-$ of the lower half-plane, we can evaluate the integrals by using the residue theorem once again, but now at the poles on the imaginary axis.

This computation yields the following result for the thermal propagator

$$G_T(\boldsymbol{x},\tau)=\int\frac{d^dp}{(2\pi)^d}\frac{\coth\left(\dfrac{\sqrt{\boldsymbol{p}^2+m^2}}{2T}\right)}{2\sqrt{\boldsymbol{p}^2+m^2}}e^{-|\tau|\sqrt{\boldsymbol{p}^2+m^2}}e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \tag{5.215}$$

This expression applies to the regime $\tau \ll \beta = 1/T$ in which quantum fluctuations play a dominant role. After a little algebra, we can now write the thermal propagator as

$$\begin{aligned}G_T(\boldsymbol{x},\tau)=&\int\frac{d^dp}{(2\pi)^d}\frac{e^{-|\tau|\sqrt{\boldsymbol{p}^2+m^2}}e^{i\boldsymbol{p}\cdot\boldsymbol{x}}}{2\sqrt{\boldsymbol{p}^2+m^2}}\\&+\int\frac{d^dp}{(2\pi)^d}\frac{1}{\exp\left(\dfrac{\sqrt{\boldsymbol{p}^2+m^2}}{T}\right)-1}\frac{e^{-|\tau|\sqrt{\boldsymbol{p}^2+m^2}}e^{i\boldsymbol{p}\cdot\boldsymbol{x}}}{\sqrt{\boldsymbol{p}^2+m^2}}\end{aligned} \tag{5.216}$$

By inspection of Eq.(5.216), we see that the first term on the r.h.s is the $T \to 0$ limit, and that it is just (as it should be!) the propagator in $D = d+1$ Euclidean space-time dimensions $G_E^{(0)}(\boldsymbol{x},\tau)$, after an integration over frequencies. The second term in the r.h.s. of Eq.(5.216) describes the contributions to the thermal propagator from thermal fluctuations, shown in the form of the Bose occupation numbers, the Bose-Einstein distribution,

$$n(\boldsymbol{p}, T) = \frac{1}{\exp\left(\frac{\sqrt{\boldsymbol{p}^2 + m^2}}{T}\right) - 1} \tag{5.217}$$

The appearance of the Bose-Einstein distribution was to be expected (and, in fact, required) since the excitations of the scalar field are bosons.

Finally, we can find the time-ordered propagator, in real time $x_0$, at finite temperature $T$, that we will denote by $G^{(0)}(\boldsymbol{x}, x_0; T)$. By means of the analytic continuation $\tau \to i x_0$ of the of the thermal propagator of Eq.(5.216), we find,

$$G^{(0)}(\boldsymbol{x}, x_0; T) = G_M^{(0)}(x) + \int \frac{d^d p}{(2\pi)^d} \frac{1}{\exp\left(\frac{\sqrt{\boldsymbol{p}^2 + m^2}}{T}\right) - 1} \frac{e^{-i|x_0|\sqrt{\boldsymbol{p}^2 + m^2}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}}}{\sqrt{\boldsymbol{p}^2 + m^2}} \tag{5.218}$$

where $G_M^{(0)}(x)$ is the (Lorentz-invariant) Minkowski space-time propagator in $D$ dimensions (i.e. at zero temperature), given in Section 5.6.4. Notice that the finite temperature contribution is not Lorentz-invariant. This result was expected since at finite temperature space and time do not play equivalent roles.

## Exercises

5.1 Path Integral for a particle in a double well potential

Consider a particle with coordinate $q$, mass $m$ moving in the one-dimensional double well potential $V(q)$

$$V(q) = \lambda \left(q^2 - q_0^2\right)^2 \tag{5.219}$$

In this problem you will use the path integral methods, in imaginary time, that were discussed in class to calculate the matrix element,

$$\langle q_0, \frac{T}{2} | -q_0, \frac{-T}{2} \rangle = \langle q_0 | e^{-\frac{1}{\hbar} HT} | -q_0 \rangle \tag{5.220}$$

to *leading order* in the semiclassical expansion, in the limit $T \to \infty$.

1 Write down the expression of the imaginary time path integral that is appropriate for this problem. Write an explicit expression for the Euclidean Lagrangian (the Lagrangian in imaginary time). How does it differ from the Lagrangian in real time? Make sure that you specify the initial and final conditions. Do not calculate anything yet!

2 Derive the Euler-Lagrange equation for this problem (always in imaginary time). Compare it with the equation of motion in real time. Find the explicit solution for the trajectory (in imaginary time) that satisfies the initial and final conditions. Is the solution unique? Explain. What is the physical interpretation of this trajectory and of the amplitude?

   **Hint**: Your equation of motion in imaginary time looks like a funny. A simple way to solve for the trajectory that you need in this problem is to think of this equation of motion as if imaginary time was real time, then to find the analog of the classical energy and to use the conservation of energy to find the trajectory.

3 Compute the imaginary time action for the trajectory you found above.

4 Expand around the solution you found above. Write a formal expression of the amplitude to leading order. Find an explicit expression for the operator that enters in the fluctuation determinant.

5.2 Path Integral for a charged particle moving on a plane in the presence of a perpendicular magnetic field.

Consider a particle of mass $m$ and charge $-e$ moving on a plane in the presence of an external uniform magnetic field perpendicular to the plane and with strength $B$. Let $\boldsymbol{r} = (x_1, x_2)$ and $\boldsymbol{p} = (p_1, p_2)$ represent