Physics 583, Academic Year 2023/2024
Spring Semester 2022
Professor Eduardo Fradkin

Problem Set No. 1
Due Date: Friday February 9, 2024, 9:00 pm US Central Standard Time

Note: You must upload a clearly legible pdf file of your solutions (LaTeX us strongly preferred) using the uploads link for Physics 583 in My.Physics

1 Vertex Functions, Effective Potential and Ward Identities

In this problem you will study the effects of interactions on the physical properties of a system of interacting relativistic fermions in $1+1$ dimensions known as the chiral Gross-Neveu model. This model is a reasonable description of the physics in quasi-one-dimensional systems, and is also of interest to investigate the behavior of quantum field theories of relativistic Fermi fields.

In $1+1$ dimensions the Dirac fields $\psi_{\alpha a}(x)$ are two-component spinors of the form

$$\psi_a = \begin{pmatrix} \psi_{R,a} \\ \psi_{L,a} \end{pmatrix} \tag{1}$$

The upper component, $\psi_{R,a}$, is the right-moving component of the fermion, and the lower component $\psi_{L,a}$ is the left-moving component of the fermion. In what follows we will consider the case in which there are N species of Dirac fermions labeled by an index $a = 1, \ldots, N$. The greek index $\alpha = R, L$ denotes the two components of the Dirac spinor in 1+1 dimensions.

Next, we define the following set of two-dimensional Dirac γ matrices

$$\gamma_0 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma_1 = i\,\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \gamma_5 = \gamma_0\gamma_1 = \,-\,\sigma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2}$$

with the notation

$$\partial\!\!\!/ = \partial_\mu \gamma^\mu = \gamma_0 \partial_0 - \gamma_1 \partial_1 \tag{3}$$

where $\mu = 0, 1$ denote the indices of a $1+1$-dimensional Minkowski space-time.

Let us introduce the operators

$$\bar{\psi}\psi = \psi^\dagger \gamma_0 \psi \equiv \psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R \tag{4}$$

and

$$\bar{\psi}\gamma_5\psi = \psi^\dagger \gamma_1 \psi \equiv \psi_R^\dagger \psi_L - \psi_L^\dagger \psi_R \tag{5}$$

Using this notation, the Lagrangian density of the chiral Gross-Neveu model is (we set the speed of light to $c = 1$)

$$\mathcal{L} = \bar{\psi}_a(x) i\partial\!\!\!/ \psi_a(x) + \frac{g}{2}\left[(\bar{\psi}_a(x)\psi_a(x))^2 - (\bar{\psi}_a(x)\gamma^5\psi_a(x))^2\right] \tag{6}$$

where we have not written down the spinor indices explicitly. In all the sections which follow below you have to use path-integral methods.

1. Derive an expression for the free Feynman propagator for the massless Dirac field in 1+1 dimensions in momentum space.

2. Derive the Feynman rules for a perturbation expansion of the fermion two-point function $S^{\alpha,\beta}_{a,b}(p)$, with $p = (p^0, p^1)$, in powers of the coupling constant g. Here α and β are the spinor indices in 1+1 dimensions and the indices a and b take values from 1 to N.

3. Derive an expression for the quantities listed below up to, and including, their second order corrections, i.e. $O(g^2)$. Draw a Feynman diagram for each contribution. Check the cancellation of the vacuum diagrams. Give a consistent sign to each contribution. **Do not do the integrals!**

   (a) the fermion two-point function

   (b) the effective coupling constant g. What correlation function should you consider: a connected Green function or a one-particle irreducible vertex function? Justify your answer.

4. (a) Show that the Lagrangian of the chiral Gross-Neveu model is invariant under the continuous *global* chiral symmetry

   $$\psi_{\alpha a}(x) \to \psi'_{\alpha a} = \left(e^{i\theta\gamma_5}\right)_{\alpha\beta}\psi_{\beta a} \tag{7}$$

   (b) Find the transformation law obeyed by the operators

   $$\hat{\Delta}_0 \equiv \bar{\psi}_a\psi_a, \qquad \hat{\Delta}_5 \equiv i\bar{\psi}_a\,\gamma_5\psi_a$$

   under this global symmetry.

   (c) Give a physical interpretation of this symmetry. What is the meaning of the operators $\hat{\Delta}_0$ and $\hat{\Delta}_5$ in terms of the right and left moving components of the fermions?.

5. Now we add the following terms to the Lagrangian the break the chiral symmetry explicitly:

   $$\mathcal{L}_{\text{chiral}} = H_0(x)\hat{\Delta}_0(x) + H_5(x)\hat{\Delta}_5(x) \tag{8}$$

   where $H_0(x)$ and $H_5(x)$ are the symmetry breaking fields. Consider now the path integral for this problem in the presence of the symmetry breaking terms. Assume that the operators $\hat{\Delta}_0$ and $\hat{\Delta}_5$ have uniform expectation values given by $\bar{\Delta}_0$ and $\bar{\Delta}_5$ respectively. Find the transformation law obeyed by the symmetry breaking fields H_0 and H_5 under a global chiral transformation of the Fermi fields by an angle θ.

6. Derive the chiral Ward Identity for the generating functional of the vertex functions for the operators $\bar{\Delta}_0$ and $\bar{\Delta}_5$.

7. Derive a Ward Identity that relates the vertex functions Γ_{00} and Γ_{55} in the limit $p \to 0$ (where $p \equiv p_\mu$). Assume that, as $H_0 \to 0$ and $H_5 \to 0$, only $\bar{\Delta}_0$ has a non zero expectation value, i.e. that the chiral symmetry is spontaneously broken.

8. Under the assumption that $\bar{\Delta}_0 \neq 0$, is there a Goldstone boson in this system? Justify your answer. Explain the significance of your answer to the original fermion problem. What does it say about its two-particle spectrum? And of the one-particle spectrum?.

Comment: In this problem you were not required to do any integrals. Therefore you cannot determine if the chiral symmetry is spontaneously broken or not. In problem set 3 we'll come back to this problem. The actual meaning of the Goldstone boson in 1+1 dimensions is subtle and we will return to this question in that problem set.

2 Renormalization of the $O(N)$ scalar ϕ^4 theory

Consider the ϕ^4 theory in a regime in which the renormalized mass is non-zero for an N-component real scalar field ϕ_a where $a = 1, \ldots, N$, with an $O(N)$-invariant Lagrangian density in D Euclidean dimensions of the form

$$\mathcal{L} = \frac{1}{2} \left(\partial_\mu \phi_a\right)^2 + \frac{m_0^2}{2} \phi_a^2 + \frac{\lambda}{4!} \left(\phi_a^2\right)^2 \tag{9}$$

As usual, repeated indices are summed over.

In the symmetric theory, *i.e.* in the absence of spontaneous symmetry breaking, the two-point and four-point 1PI vertex functions $\Gamma^{(2)}_{ab}(p)$ and $\Gamma^{(4)}_{abcd}(p_1, \ldots, p_4)$ take the symmetric form

$$\Gamma^{(2)}_{ab}(p) = \delta_{ab}\, \bar{\Gamma}^{(2)}(p) \tag{10}$$

and

$$\Gamma^{(4)}_{abcd}(p_1, \ldots, p_4) = S_{abcd}\, \bar{\Gamma}^{(4)}(p_1, \ldots, p_4) \tag{11}$$

where

$$S_{abcd} = \frac{1}{3} \left(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right) \tag{12}$$

1. Find all the contributions to $\Gamma^{(2)}$ and $\Gamma^{(4)}$ to **one loop order** for this N-component theory. Write down and to draw all Feynman diagrams and their associated analytic expressions. Write your results in terms of the integrals discussed in class (for the one-component theory). Do not do the integrals. Derive the explicit dependence of each diagram on the number of components N.

2. Define a set of *renormalization conditions*, at zero external momentum, for the vertex functions $\bar{\Gamma}^{(2)}(p)$ and $\bar{\Gamma}^{(4)}(p_1, \ldots, p_4)$ in the symmetric massive theory.

3. Determine m_0^2, and λ in terms of the renormalized mass μ and coupling constant g at fixed momentum cutoff Λ, **to one loop order**. Express your answers in terms of the integrals defined in class. **Do not do the integrals!**.

4. Show that the renormalization conditions of part 2) and the renormalization constants obtained in part 3) yields finite vertex functions at arbitrary values of the external momentum, **to one loop order** in perturbation theory.

Useful identities:

$$\sum_c S_{abcc} = \frac{N+2}{3}\,\delta_{ab}$$

$$\sum_{ij} S_{abij} S_{ijcd} = \frac{2}{3}\,S_{abcd} + \frac{N+2}{9}\,\delta_{ab}\delta_{cd}$$