Physics 583
Advanced Quantum Field Theory
Academic Year 2023/2024
Spring Semester 2024
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Problem Set No. 2
Introduction to the Renormalization Group
Due Date: Sunday March 17, 2021, 5:00 pm US Central Time

1 Renormalization Group for the Ashkin-Teller Model

The Ashkin-Teller Model is a simple generalization of the Ising Model in which there are two Ising degrees of freedom, σ and τ, each taking values ± 1, defined on the sites of a lattice. Here we will consider a D-dimensional hypercubic lattice whose sites are labelled by the two-component lattice vectors $\boldsymbol{r}$. The interactions are restricted to nearest neighboring sites and the classical Hamiltonian H is

$$H = -K_2 \sum_{\langle \boldsymbol{r}, \boldsymbol{r}\,' \rangle} (\sigma(\boldsymbol{r})\sigma(\boldsymbol{r}\,') + \tau(\boldsymbol{r})\tau(\boldsymbol{r}\,')) - K_4 \sum_{\langle \boldsymbol{r}, \boldsymbol{r}\,' \rangle} \sigma(\boldsymbol{r})\sigma(\boldsymbol{r}\,')\tau(\boldsymbol{r})\tau(\boldsymbol{r}\,') \quad (1)$$

where $\langle \boldsymbol{r}, \boldsymbol{r}\,' \rangle$ denotes nearest neighboring sites; $K_2 > 0$ and $K_4 > 0$ are two coupling constants. Clearly for $K_4 = 0$ this system is equivalent to two decoupled Ising models.

1. Show that the following statements are true:
   (a) If $K_2 \to \infty$, the system is in its ground state, and find the possible ground state(s).
   (b) If $K_4 \to \infty$ the system is equivalent to a *single* Ising model with coupling constant $K = 2K_2$
   (c) If $K_2 \to 0$ the system is also equivalent to a *single* Ising model with coupling constant K_{eff}. Find K_{eff}.
2. Consider now the Ashkin-Teller model on a *one-dimensional* lattice. Use a decimation method (analogous to what was done in class for the one-dimensional Ising model) to derive an exact renormalization group transformation. Find the fixed points, and calculate their eigenvalues. Draw the RG flows explicitly. Discuss the similarities and differences with the Ising model

3. Consider now the Ashkin-Teller model on a D-dimensional hypercubic lattice. Use the Migdal-Kadanoff bond moving procedure to derive a block-spin transformation for $D = 1 + \varepsilon$, where $\varepsilon > 0$ and "small".

   (a) Find *all* the fixed points, both stable and unstable. Calculate *all* the eigenvalues of the RG transformation for each fixed point, and the correlation length exponent ν for each fixed point.

   (b) Sketch the qualitative RG flows on the $K_2 - K_4$ plane. Use the flows to derive a phase diagram. What is the order parameter(s) which labels each phase? Justify your arguments.

   Warning: The Midgal-Kadanoff bond-moving procedure yields an unphysical phase transition at $K_2 \to \infty$, with $K_4 < \infty$. It is easy to see that the associated stable fixed points are physically equivalent.

2 Momentum-Shell RG for the Non-Linear σ-Model

In this problem you will be asked to consider the $O(N)$ non-linear σ-model, in $D = 2 + \varepsilon$ dimensions, coupled to an external field. Since we will consider this theory in Euclidean space it is equivalent to a classical Heisenberg model.

We will parametrize the configurations by means of an N-component field $\boldsymbol{n}(\boldsymbol{x})$, which satisfies the constraint $\boldsymbol{n}^2 = 1$. The Euclidean Lagrangian (or classical Hamiltonian) is $\mathcal{L}$

$$\mathcal{L} = \frac{1}{2g}(\partial_\mu \boldsymbol{n}(\boldsymbol{x}))^2 + \frac{1}{2g}\boldsymbol{H}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}) \tag{2}$$

The partition function Z is

$$Z = \int \mathcal{D}\boldsymbol{n}(\boldsymbol{x}) \prod_{\boldsymbol{x}} \delta(\boldsymbol{n}^2(\boldsymbol{x}) - 1)\, e^{-\int d^d x\, \mathcal{L}} \tag{3}$$

We are going to follow the procedure to decompose the field $\boldsymbol{n}(\boldsymbol{x})$ into its *slow* and *fast* components. Let $\boldsymbol{n}_0(\boldsymbol{x})$ be an N-component *slowly* varying configuration of the σ-model which is *also* a solution of the classical equations of motion (*i.e.,* it extremizes the Euclidean action). Also, let $\phi_i(\boldsymbol{x})$ ($i = 1, \ldots, N-1$.) be an (essentially unrestricted) $N-1$-component field which we will use to parametrize the fluctuations of the field as follows. Let $\{\boldsymbol{e}_i(\boldsymbol{x})\}$ be a set of $N-1$ unit-length vectors defined at each point in space. They are required to form, together with $\boldsymbol{n}_0(\boldsymbol{x})$, an orthonormal basis for the local configurations, *i.e.,*

$$n_0^a(\boldsymbol{x}) e_i^a(\boldsymbol{x}) = 0; \qquad e_i^a(\boldsymbol{x}) e_j^a(\boldsymbol{x}) = \delta_{i,j} \tag{4}$$

for $i, j = 1, \ldots, N-1$ and $a = 1, \ldots, n$ (repeated indices are summed over). Thus, we can write

$$n^a(\boldsymbol{x}) = \sqrt{1 - \boldsymbol{\phi}^2(\boldsymbol{x})}\ \ n_0^a(\boldsymbol{x}) + \sum_{i=1}^{N-1} \phi_i(\boldsymbol{x}) e_i^a(\boldsymbol{x}) \tag{5}$$

In other terms, we have defined a *local frame*. The local changes of the slow field $n_0^a(x)$ and of the unit vectors e_i^a (*i.e.,* of the local frame) can also be expanded in that basis

$$\partial_\mu n_0^a(\boldsymbol{x}) = \sum_{i=1}^{N-1} B_\mu^i(\boldsymbol{x}) e_i^a(\boldsymbol{x}) \tag{6}$$

$$\partial_\mu e_i^a(\boldsymbol{x}) = \sum_{j=1}^{N-1} A_\mu^{ij}(\boldsymbol{x}) e_j^a(\boldsymbol{x}) - B_\mu^i(\boldsymbol{x}) n_0^a(\boldsymbol{x}) \tag{7}$$

where

$$B_\mu^i(\boldsymbol{x}) = \boldsymbol{e}_i(x) \cdot \partial_\mu \boldsymbol{n}_0(x), \qquad A_\mu^{ij}(x) = -A_\mu^{ji}(\boldsymbol{x}) = -\boldsymbol{e}_i(x) \cdot \partial_\mu \boldsymbol{e}_j(x) \tag{8}$$

1. Write the coupling constant g and the magnetic field $\boldsymbol{H}$ in terms of the dimensionless coupling constant u and dimensionless field $\boldsymbol{h}$, by using dimensional analysis in terms of a momentum cutoff Λ.

2. Write the action as a function(al) of the fluctuating field $\boldsymbol{\phi}(x)$ in the background of the field, slow configuration $\boldsymbol{n}_0(x)$, parametrized by the fields $B_\mu^i(x)$ and $A_\mu^{ij}(x)$. Expand the Euclidean action as a power series expansion of the field $\boldsymbol{\phi}(x)$, and write down the explicit for of the linear and quadratic terms in $\phi_i(x)$.

3. Show that for the configuration $\boldsymbol{n}_0(\boldsymbol{x})$ to be stationary, the background fields $B_\mu^i(x)$ and $A_\mu^{ij}(x)$ must satisfy the equation of motion

$$D_\mu^{ij} B_\mu^j(x) \equiv \left[\partial_\mu \delta_{ij} - A_\mu^{ij}(x)\right] B_\mu^j(x) = 0 \tag{9}$$

   Hint: Demand that the action is stationary with respect to a variation of the fluctuation field ϕ_i.

4. Use the momentum shell integration technique to integrate-out the fast variables of the field $\phi_i(x)$ inside the momentum shell $b\Lambda \leq |\boldsymbol{p}| \leq \Lambda$, with $b = 1 - \delta\ell \lesssim 1$ and $\delta\ell$ small. Obtain a set of differential equations which govern the (differential) renormalization for u and $h = |\boldsymbol{h}|$.

5. Find the fixed points and draw the qualitative flows for dimension $D = 2 + \varepsilon$. Calculate the eigenvalues of u and h at each fixed point and determine when are u and h relevant, irrelevant or marginal. Show that precisely *at* $D = 2$ there is a marginal operator.

6. Use a renormalization group argument to find an analytic expression for the singular dependence of the correlation length as a function of u for $u > u_c$ and $h = 0$. Express your result in terms of the critical exponent ν.

7. Use a renormalization group argument to find the behavior of the *order parameter*, the *spontaneous magnetization* $\mathcal{M}^a = \langle n^a(\boldsymbol{x})\rangle$, as a function of u at $h = 0$ and *close* to u_c. Show that it obeys a power law of the form $\mathcal{M} \sim (u_c - u)^\beta$ and compute the critical exponent β to lowest order in ε. Use a similar RG argument to find the field dependence of the magnetization *at* u_c and determine the value of the critical exponent δ of the h dependence of $\mathcal{M}$ as $h \to 0$, $\mathcal{M} \sim |h|^{1/\delta}$

8. Consider the special (and important) case of $D = 2$ dimensions.

   (a) Solve the RG flow equation for the dimensional coupling constant $u = u(\Lambda)$ as an explicit function of the cutoff Λ in terms of its value at a reference scale μ.

   (b) Discuss the behavior of this effective (or *running*) coupling constant in the ultra-violet regime $\Lambda \gg \mu$. Does the running coupling constant get big or small? How fast? Why is this behavior referred to as *asymptotic freedom*?

   (c) Find the behavior of the correlation length ξ as a function of the dimensional coupling constant u for $D = 2$. Be explicit about the assumptions you make and what is their physical justification.

NOTE: In this problem you <u>have</u> to do the momentum-shell integral. You may find it useful to express your angular integrations in terms of the area of the unit hypersphere in D dimensions

$$S_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}$$

where $\Gamma(z)$ is the Euler Gamma function,

$$\Gamma(z) = \int_0^\infty dt\, t^{z-1} e^{-t}$$

which is convergent for Re $z > 0$ and where $\Gamma(1) = 1$.