Physics 583
Advanced Quantum Field Theory
Academic Year 2023/2024
Spring Semester 2024
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Problem Set No. 3
The Renormalization Group
Due Date: Friday April 12, 2024, extended to
Tuesday April 16, 2024

1 Renormalization of an Interacting Field Theory of Fermions

In another problem set we discussed the chiral Gross-Neveu model, which is a simple model of chiral symmetry breaking in particle physics, and of charge density waves in condensed matter. For simplicity we will consider the case of $1 + 1$-dimensional space-time although it is easy to work out a generalization to higher dimensions. Most of the questions below are formulated for the theory in Minkowski space-time. Naturally, you will have to rotate the theory to Euclidean space-time to do the integrals and to derive the RG equations.

The Lagrangian density of the Chiral Gross-Neveu Model is

$$\mathcal{L} = \bar{\psi}_a i \partial\!\!\!/ \psi_a + \frac{g_0}{2N} \left(\left(\bar{\psi}_a \psi_a \right)^2 - \left(\bar{\psi}_a \gamma_5 \psi_a \right)^2 \right) \tag{1}$$

where ψ_a is a two-component Dirac spinor

$$\psi_a(x) \equiv \begin{pmatrix} R_a \\ L_a \end{pmatrix} \tag{2}$$

with R_a an L_a being the amplitudes for the (chiral) Right and Left fields respectively, with $a = 1, \ldots, N$. In this problem we are going to assume that N is so large that the limit $N \to \infty$ is a reasonable approximation. We will use the same basis for the spinors as in Problem Set 3. In this basis, the two-dimensional γ-matrices are given in terms of Pauli matrices: $\gamma_0 = \sigma_1$, $\gamma_1 = i\sigma_2$ and $\gamma_5 = -\sigma_3$. Recall the notation: $\partial\!\!\!/ = \partial_\mu \gamma^\mu = \gamma_0 \partial_0 - \gamma_1 \partial_1$.
Note: here we defined a coupling constant g_0 which differs from the coupling constant g we defined earlier just by a scale factor, $g = 2g_0/N$.

1. The Lagrangian of this system contains an interaction term which is quartic in the Fermi fields. Instead of using straightforward perturbation theory you will study this system in the large N limit. In order to do that

you first need to verify the following Gaussian identity, also known as a Hubbard-Stratonovich transformation:

$$\int \mathcal{D}\sigma(x) \, e^{i \int d^2x \left[-\frac{\sigma^2(x)}{2} - \sqrt{\frac{g_0}{N}} \bar{\psi}(x)\psi(x)\sigma(x) \right]} = \mathcal{N} e^{i \int d^2x \frac{g_0}{2N} (\bar{\psi}\psi)^2}$$

(3)

where $\mathcal{N}$ is a suitable normalization constant, and

$$\bar{\psi}\psi \equiv \sum_{a=1}^{N} \bar{\psi}_a(x)\psi_a(x)$$

(4)

2. Use an identity of the type of the one derived in 1), involving two scalar fields $\sigma(x)$ and $\omega(x)$, to write the Lagrangian of the Chiral Gross-Neveu model in a form which is *quadratic* in the Fermi fields.

3. In Problem Set 3 you found that this model is invariant under the continuous global chiral transformation $\psi_a = e^{i\,\theta\gamma_5}\psi'_a$. What transformation law should the scalar fields σ and ω satisfy?. How are these fields related to the operators Δ_0 and Δ_5 of Problem Set 3?.

4. Integrate out the Fermi fields and find the effective action for the scalar fields σ and ω. Watch for the factors of N and be careful with the signs!. By an appropriate rescaling of the scalar fields show that the effective action has the form $S_{\text{eff}} = N\bar{S}$. Determine the form of $\bar{S}$.

5. Now you will consider the limit $N \to \infty$. Find the equation of *Saddle-Point Equations* which determine the average values of the scalar fields in this limit. Find the solution of the Saddle-Point Equations with lowest energy. Is the solution unique?. *In order to compute the integrals, first do the traces over the Dirac indices. The resulting expressions involve integrals that you should Wick rotate to the Euclidean domain. To regularize the integrals use dimensional regularization in $2 + \epsilon$ dimensions.* What quantities need to be renormalized in order to make the Saddle-Point Equations finite?. How many renormalization constants do you need?. Give your answers in terms of coupling constant and wave function renormalizations. Be careful to include the dependence in the dimensionality $2 + \epsilon$. Determine the renormalization constants using the *minimal subtraction* scheme.

6. Compute the β-function for the dimensionless renormalized coupling constant t you derived in section 1.5. Find its fixed points and flows in $1 + 1$ dimensions. Solve the differential equation $\beta(t) = \kappa \frac{\partial t}{\partial \kappa}$, where κ is the momentum scale. Determine the asymptotic behavior of t in the limit $\kappa \to \infty$. Is the interaction term relevant, irrelevant or marginal?.

7. Use the results of the previous sections to write the Saddle-Point Equations in term of renormalized quantities alone. In particular, find the

dependence of the average values of the scalar fields on the renormalized coupling constant.

8. Consider now the Fermion propagator in the $N \to \infty$ limit. Are the Fermions massive or massless?. If the former is true, what is the value of the Fermion mass and how does it relate to the expectation values of the scalar fields?.

9. Find the *renormalized* effective action for the scalar fields σ and ω to *leading order* in the $\frac{1}{N}$ expansion, i.e. to order $\frac{1}{N}$. Determine the renormalized propagator of the scalar fields at this order. Are the scalar fields massive, or massless?.

10. Consider now the effect of a field that breaks the chiral symmetry. The extra term in the Lagrangian is $\mathcal{L}_{\text{sources}}$ given by

$$\mathcal{L}_{\text{sources}} = H_0(x)\bar{\psi}_a(x)\psi_a(x) + H_5(x)\bar{\psi}_a(x)\gamma_5\psi_a(x) \qquad (5)$$

Find the new effective action of this theory in the presence of these symmetry breaking fields. Derive the modified Saddle-Point Equations. Solve the new Saddle-Point Equations for the case $H_0(x) = H$ and $H_5(x) = 0$.

11. Repeat the renormalization procedure employed for the theory without sources, now for the case with sources present. Be careful to include a wave function renormalization. Derive the renormalized Saddle-Point Equations. Renormalize the propagators as in section 9.

12. By functionally differentiating the path integral with respect to the sources derive a equation of identities which relate expectation values of the scalar fields σ and ω to expectation values of the Fermion bilinears $\bar{\psi}\psi$ and $\bar{\psi}\gamma_5\psi$. In particular find a formula which relates the propagators of σ and ω to the propagators of the Fermion bilinears.

13. Use the Ward Identity you derived in Problem Set 1 to derive a relation between the two point functions of the scalar fields at zero momentum, and the external symmetry breaking field. Do the results you found in section 9 satisfy these relations?.

14. Derive the renormalization group equations (Callan-Symanzik) satisfied by the scalar fields in the absence of external sources. Solve these Callan-Symanzik equations in terms of a momentum rescaling factor ρ and a *running* coupling constant.
Note: Unlike renormalized perturbation theory, here you found a solution of the RG equations that holds *for all* values of the coupling constant. This is possible because of the large N limit, which is non-perturbative in the coupling constant.

15. Use the solutions of section 14 to find the asymptotic behavior of the two point functions of the scalar fields at large momenta.

Useful Formulas (Euclidean metric)

Let A and B be two *operators* (or matrices), then

$$\operatorname{Tr} \ln (A + B) = \operatorname{Tr} \ln A + \operatorname{Tr} \ln \left(I + A^{-1}B\right)$$

and

$$\ln(1 + x) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$$

where A^{-1} is the inverse of the operator A, and I is the identity operator.
Hint: you can use the following identities to derive your expansion:

$$\operatorname{Tr} \ln \left(i\slashed{\partial} + M + \delta M\right) = \operatorname{Tr} \ln \left(i\slashed{\partial} + M\right) + \operatorname{Tr} \left(I + \left(i\slashed{\partial} + M\right)^{-1} \delta M\right)$$

and

$$\operatorname{Tr} \left(I + \left(i\slashed{\partial} + M\right)^{-1} \delta M\right) = \int d^2 x \sum_{\alpha\beta} S_{\alpha\beta}(x, x) \delta M_{\beta\alpha}(x, x)$$

$$- \frac{1}{2} \int d^2 x \int d^2 x' \sum_{\alpha\beta\alpha'\beta'} S_{\alpha\beta}(x, x') \delta M_{\beta\alpha'}(x', x') S_{\alpha'\beta'}(x', x) \delta M_{\beta'\alpha}(x, x) + \dots$$

where $S_{\alpha\beta}(x, y)$ is the Dirac propagator

$$S_{\alpha\beta}(x, y) = \langle x, \alpha | \frac{1}{i\slashed{\partial} + M} | y, \beta \rangle$$

where M is a (diagonal) operator.

Useful integral:

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 + 2\boldsymbol{p} \cdot \boldsymbol{q} + m^2)^{\alpha}} = \frac{1}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(\alpha - \frac{D}{2})}{\Gamma(\alpha)} (m^2 - p^2)^{\frac{D}{2} - \alpha}$$

where $\Gamma(z)$ is the Gamma function

$$\Gamma(z) = \int_0^{\infty} dt \, t^z e^{-t}$$

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2 The free massless scalar field fixed point in $D = 2$ space-time dimensions

Consider a free massless scalar field $\phi(x)$ in $D = 2$ Euclidean dimensions, $\boldsymbol{x} = (x_1, x_2)$. This system is known in Statistical Mechanics as the Gaussian Model. The (Euclidean) Lagrangian density is

$$\mathcal{L} = \frac{K}{2} \, (\nabla \phi)^2 \tag{6}$$

where K is a positive real parameter (a "coupling constant"). The Euclidean propagator for this system in a region of linear size $L \equiv m^{-1}$ (a very large disk) is

$$G(\boldsymbol{x} - \boldsymbol{x}') = \langle \phi(\boldsymbol{x})\phi(\boldsymbol{x}') \rangle = \frac{1}{K} \int \frac{d^2 p}{(2\pi)^2} \frac{e^{i\boldsymbol{p} \cdot (\boldsymbol{x} - \boldsymbol{x}')}}{p^2 + m^2} \tag{7}$$

This formula is valid for $|\boldsymbol{x} - \boldsymbol{x}'| \ll m^{-1} = L$ where the propagator takes the form

$$G(\boldsymbol{x} - \boldsymbol{x}') = -\frac{1}{4\pi K} \ln \left(\frac{|\boldsymbol{x} - \boldsymbol{x}'|^2 + a^2}{L^2} \right) \tag{8}$$

where a is a short distance cutoff. You will use this propagator in the rest of this problem.

1. *Scale Invariance*: Show that $S[\phi]$ invariant under scale transformations (dilations)

$$\boldsymbol{x} \to \boldsymbol{x}' = \lambda \boldsymbol{x} \tag{9}$$

   where $\lambda > 0$ is a positive real number.

2. *Operators*: Consider the *vertex* operators $V_q(x) = e^{iq\phi(x)}$, where q is an *integer*. Add a source term to the action of the form

$$J(\boldsymbol{x}) = i \sum_{j=1}^{N} q_j \delta^{(2)}(\boldsymbol{x} - \boldsymbol{x}_j) \tag{10}$$

   to show that the correlation function of N vertex operators

$$\langle V_{q_1}(\boldsymbol{x}_1) \ldots V_{q_N}(\boldsymbol{x}_N) \rangle \tag{11}$$

   is invariant under the shift $\phi(x) \to \phi(x) + \alpha$ (with α arbitrary) only if $\sum_{j=1}^{N} q_j = 0$.

3. *Correlators*: Show that the correlation functions of N vertex operators with "charges" $q_1, \ldots, q_N$ are given by

$$\langle V_{q_1}(\boldsymbol{x}_1) \ldots V_{q_N}(\boldsymbol{x}_N) \rangle = e^{-\left(\sum_{j=1}^{N} q_j \right)^2 G(0) - \sum_{j>j'=1}^{N} q_j q_{j'} [G(\boldsymbol{x}_j - \boldsymbol{x}_{j'}) - G(0)]} \tag{12}$$

   Show that, in the $L \to \infty$ limit, these correlation functions vanish identically unless the "charge neutrality" condition, $\sum_{j=1}^{N} q_j = 0$ is satisfied.

4. *Two Point Function*: Use the expression you just derived to compute the explicit form of the correlation function of two vertex operators

$$G_q^{(2)}(\boldsymbol{x} - \boldsymbol{x}') = \langle V_q(\boldsymbol{x})V_{-q}(\boldsymbol{x}')\rangle \tag{13}$$

Show that the scaling dimension Δ_q of the vertex operator V_q is

$$\Delta_q = \frac{q^2}{4\pi K} \tag{14}$$

5. *Three Point Function*: Show that the correlation function of three vertex operators $V_{q_1}(\boldsymbol{x}_1)$, $V_{q_2}(\boldsymbol{x}_2)$ and $V_{q_3}(\boldsymbol{x}_3)$ is given by

$$\langle V_{q_1}(\boldsymbol{x}_1)V_{q_2}(\boldsymbol{x}_2)V_{q_3}(\boldsymbol{x}_3)\rangle = \frac{1}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|^{\Delta_{12}}|\boldsymbol{x}_2 - \boldsymbol{x}_3|^{\Delta_{23}}|\boldsymbol{x}_3 - \boldsymbol{x}_1|^{\Delta_{31}}} \tag{15}$$

with $q_1 + q_2 + q_3 = 0$. Show that $\Delta_{12} = \Delta_{q_1} + \Delta_{q_2} - \Delta_{q_3}$, $\Delta_{23} = \Delta_{q_2} + \Delta_{q_3} - \Delta_{q_1}$ and $\Delta_{31} = \Delta_{q_3} + \Delta_{q_1} - \Delta_{q_2}$.

6. *Scaling Dimensions*: Suppose we were to add to the free field fixed point Lagrangian a perturbation of the form

$$\mathcal{L}_{\text{int}} = g\cos(q\phi(x)) = \frac{g}{2}\ (V_q(x) + V_{-q}(x)) \tag{16}$$

where g is a coupling constant. and $q \in \mathbb{Z}$. Determine the values of K for which values of q this perturbation is a) marginal, b) relevant and c) irrelevant. Use these results to write down the renormalization group β-function $\beta(g)$ to *linear order* in the coupling constant g.

7. *Operator Product Expansion* (OPE).

(a) Show that the operators $\{V_{q_i}(x)\}$ obey an OPE of the form

$$V_{q_1}(x_1)V_{q_2}(x_2) \sim \frac{C_{q_1,q_2,q_1+q_2}}{|x_1 - x_2|^{\mu_{q_1,q_2}}}V_{q_1+q_2}\left(\frac{x_1 + x_2}{2}\right)$$

as $x_1 \to x_2$. Use the results of the above subsections to find the coefficient C_{q_1,q_2,q_1+q_2} and the exponent μ_{q_1,q_2}.

(b) Show that the operators $V_q(x)$ and $V_{-q}(y)$ obey an OPE of the form

$$V_q(x)V_{-q}(y) \sim \frac{1}{|x - y|^{\eta_q}} + \frac{C_q}{|x - y|^{\mu_q}} : (\nabla\phi)^2\left(\frac{x + y}{2}\right):$$

as $x \to y$, where $: (\nabla\phi)^2\ (x) := (\nabla\phi)^2\ (x) - \langle(\nabla\phi)^2\ (x)\rangle$. Find the coefficient C_q and the exponents η_q and μ_q.

(c) Show that the OPE of the operator $E(x) =: (\nabla\phi)^2\ (x)$: with itself does not contain the operator $E(x)$: *i.e.* $C_{EEE} = 0$.

8. *Renormalization Group*: Let us now consider the sine-Gordon theory, whose (Euclidean) Lagrangian density is

$$\mathcal{L} = \frac{K}{2} \, (\nabla \phi)^2 + \frac{g}{2} \, (V_q(x) + V_{-q}(x)) \tag{17}$$

where K and g play the role of the coupling constants. Use the results of the OPE of the previous part to construct the RG beta-functions for the "stiffness" K (this effect can also be regarded as a wave function renormalization) and the coupling constant g, up to quadratic order in g.

Note: the free scalar field with the restriction that the only allowed operators are the vertex operators with integer-valued "charge" q is an example of a *conformal field theory*. It is known as the *compactified boson*. In string theory the field ϕ is regarded as the coordinate of a *bosonic string* on a circle of radius R. The restriction implies that the boson has a *compactification radius $R = 1$*. When $K \neq 1$ the boson has effectively a compactification radius $R = \sqrt{K}$.